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Orthometric Height Determination Using GPS Observations and the Integrated Geodesy Adjustment Model

Günter W. Hein

Rockville, MD
April 1985

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ORTHOMETRIC HEIGHT DETERMINATION USING GPS OBSERVATIONS AND THE INTEGRATED GEODESY ADJUSTMENT MODEL

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ABSTRACT. An integrated geodesy adjustment is presented for the determination of orthometric height differences using observations of the satellites of the Global Positioning System (GPS) in combination with all available terrestrial observations. In particular, linear (pseudo-) observation equations for orthometric heights and the GPS base-line components are developed which allow for the consideration of noise in both data types.

1. INTRODUCTION

Initial relative positioning results (Goad and Remondi 1984) using the satellites of the Global Positioning System (GPS) encourage users to compute orthometric height differences, $\Delta H = H_2 - H_1$, by use of the well-known relation:

$$(H_2-H_1) = (h_2-h_1) - (N_2-N_1) \tag{1-1}$$

where $\Delta h = h_2 - h_1$ is the difference in ellipsoidal heights, h_i, and $\Delta N = N_2 - N_1$ is the difference in geoid heights, N_i. Whereas Δh can be derived by GPS over distances of the order of 100 km with centimeter, or even, subcentimeter accuracy, ΔN has to be determined using other data sources that do not meet the same level of accuracy.

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This research was performed during May-August 1984, when the author was a Senior Visiting Scientist at the National Geodetic Survey, under the auspices of the Committee on Geodesy, National Research Council, National Academy of Sciences, Washington, D.C.

In general, two approaches are possible for computing precise geoid heights or geoid height differences, respectively:

(1) Gravimetric determinations using Stokes' integral or least-squares collocation.

(2) Interpolation of the relative geoid height surface from stations where the orthometric height is known by leveling, and precise ellipsoidal heights (or differences) are determined by GPS so that equation (1-1) can be used in the reverse sense.

Engelis et al. (1984) compared geoid undulation differences derived by the gravimetric determination in (1) above, with results obtained by GPS, see (2) above. The two sets of computed geoid undulation differences have an r.m.s. discrepancy of $\pm$ 5 cm for an average station separation of the order of 14 km. In spite of the fact that this is already a good result, the resultant orthometric height differences cannot replace conventional geodetic (precise) leveling, which in a third-order network for the equivalent line length yields an estimate with an uncertainty of about $\sigma = \sigma_{1km}\sqrt{14} = 7.5$ mm when assuming $\sigma_{1km} = 2.0$ mm for the United States.

However, most studies have been performed in areas having only small elevation differences and a "smooth" geoid. Therefore, error estimates would increase when applying eq. (1-1) in mountainous areas, because of the influence of the terrain on the gravimetric determinations. On the other hand, using approach (2) assumes precise vertical control has already been established by leveling in the area of consideration. Thus, the accuracy of the resulting geoid-height surface relies on the accuracy of the leveling, which is not always as good as usual error propagation indicates. Refraction and magnetic influences are two examples of systematic error sources that were not properly accounted for in the past. Cost effectiveness and reduction in observing time are important criteria in geodetic observation strategies for establishing and monitoring networks. Although geodetic leveling is one of the most accurate geodetic measurement techniques, it might not be possible to apply it in the future for monitoring large networks in and within short time intervals with reasonable costs. It is, therefore, worthwhile to concentrate again on the precise determination of geoid heights and differences, respectively, to take advantage of relation (1-1).

In the following, a new integrated geodesy adjustment model for orthometric heights is presented to overcome the difficulties and drawbacks of the models previously applied to that problem. It is not required to hold fixed any quantity beforehand to derive another one, as, for example, is done when determining the geoid height surface from leveled orthometric height and GPS-derived ellipsoidal heights. Both quantities are considered as observations in the integrated geodesy adjustment, with an associated covariance matrix. For that purpose a new linear observation equation for (orthometric) heights is developed.

2. THE PRINCIPLE OF AN INTEGRATED GEODESY ADJUSTMENT

For the reader who is not familiar with the integrated geodesy adjustment, the principle is briefly outlined here. Further details can be found in Hein (1982a,b), Hein and Landau (1983).

Every geodetic measurement, l, can be expressed as a nonlinear functional depending on one or several position vectors $\underline{x} = (x,y,z)$ in space, and on the gravity field of the Earth, symbolically written:

$$l = F(\underline{x},W) \tag{2-1}$$

where W is the gravity potential

$$W = V + \omega^2 (x^2 + y^2)/2 \quad . \tag{2-2}$$

V is the potential of the gravitational force, ω is the angular velocity of the Earth's rotation and (x,y,z) are Cartesian coordinates in a geocentric reference frame. The z-axis coincides with the rotation axis of the Earth.

As is done in usual adjustment practice, we linearize eq. (2-1) by introducing approximate values $\underline{x}^0$ and U for the position and gravity potential, respectively:

$$\underline{x} = \underline{x}^0 + \delta\underline{x} \tag{2-3}$$

$$W(\underline{x}) = U(\underline{x}) + T(\underline{x}) \quad . \tag{2-4}$$

If U is a so-called normal potential associated with an adopted reference system of the International Association of Geodesy, T is then the disturbing potential as in classical geodesy. Thus, eq. (2-1) is expanded in a Taylor series at $P(\underline{x}^o,U)$, and neglecting higher order terms one obtains:

$$l = F(\underline{x}^o,U) + \sum_{i=1}^{3} F_{x_i}(\underline{x}^o,U)\delta x_i + L(T) \qquad (2\text{-}5)$$

or, in matrix notation, including the noise, n, in the observations:

$$\delta\underline{l} = \underline{a}^T\ \delta\underline{x} + \underline{Rt} + \underline{n} \qquad (2\text{-}6)$$

where

$$a_i = F_{x_i}(\underline{x}^o,U) = \frac{\partial F}{\partial x_i}(\underline{x}^o,U), \qquad \text{and} \qquad (2\text{-}7)$$

$$\delta\underline{l} = \underline{l} - F(\underline{x}^o,U). \qquad (2\text{-}8)$$

L(T) in eq. (2-5) is a linear operator applied on the disturbing potential T. It is expressed in eq. (2-6) by a coefficient matrix $\underline{R}$ and a vector $\underline{t}$ containing the disturbing potential and its functionals.

As shown in Eeg and Krarup (1973), the linear system of eqs. (2-5) is solved by minimizing

$$\underline{n}^T\ \underline{C}_{nn}^{-1}\ \underline{n} + \underline{t}^T\ \underline{K}_{tt}^{-1}\ \underline{t} = \min \qquad (2\text{-}9)$$

and thus we obtain as an estimate for $\delta\underline{x}$ and $\underline{t}$, the corresponding formulas of a general collocation-type model (Moritz 1980:116). $\underline{C}_{nn}$ and $\underline{K}_{tt}$ are covariance matrices of $\underline{n}$ and $\underline{t}$, respectively.

The integrated geodesy adjustment can be considered as a discrete solution of a free boundary value problem, determining simultaneously both the coordinates and the functionals of the disturbing potential.

3. THE OBSERVATION EQUATIONS FOR ORTHOMETRIC HEIGHTS IN THE INTEGRATED GEODESY ADJUSTMENT

The observation equation for orthometric heights, H, in the integrated geodesy adjustment model will be derived next. It might be considered as the three-dimensional analogue of

$$H = h - N \tag{3-1}$$

or eq. (1-1), when dealing with the corresponding differences in ellipsoidal heights, h, and geoidal heights, N.

We start with the formula for orthometric height using Helmert's definition (Heiskanen and Moritz 1967:167):

$$H(P) = \frac{C(P)}{g(P) + \alpha H(P)} \tag{3-2}$$

where

- C is the geopotential difference, $C(P) = W_o(Q) - W(P)$, where $W_o(Q)$ belongs to point Q on the geoid,
- g is the actual gravity at surface point P, and
- α is a coefficient derived from the normal gravity field,

$$\alpha = -\left(\frac{1}{2}\frac{\partial\gamma}{\partial h} + 2\pi k\rho\right) \quad \text{(See Heiskanen and Moritz 1967: 167)} \tag{3-3}$$

where

- $\frac{\partial\gamma}{\partial h}$ is the normal vertical gravity gradient,
- k is the gravitational constant, and
- ρ is the normal density, 2.67 g/cm^3

Inserting numerical values in eq. (3-3), α is determined to be

$$\alpha = 0.0424 \text{ gal km}^{-1} . \tag{3-4}$$

To linearize eq. (3-2), it is divided into a so-called normal part or approximate value, H^o, and a disturbing (linear) part, δH:

$$H = H^o + \delta H . \tag{3-5}$$

Applying Taylor's theorem at $\underline{x}_p = \underline{x}_p^o$, and neglecting terms of higher order, one obtains:

$$\delta H = \frac{\delta C}{g^o + \alpha H^o} - \frac{C^o \, \delta g}{(g^o + \alpha H^o)^2} \tag{3-6}$$

where

$$\delta C = \delta W_o - \delta W_p \, . \tag{3-7}$$

In the further linearization process we introduce approximate values for gravity, δg, and potentials, δW_o and δW_p, in eq. (3-6). To be consistent with former papers (Hein 1982a,b) , the symbol j is used for normal gravity

$$g(\underline{x}_p) = g^o(\underline{x}_p^o) + \delta g(\underline{x}_p^o) = j(\underline{x}_p^o) + \delta g(\underline{x}_p^o) \, , \tag{3-8}$$

$$W_p(\underline{x}_p^o) = U(\underline{x}_p^o) + \delta W_p(\underline{x}_p^o) \qquad \text{and} \tag{3-9}$$

$$W_o(\underline{x}_Q^o) = U(\underline{x}_Q^o) + \delta W_o(\underline{x}_Q^o) \tag{3-10}$$

For the disturbing quantities we assume that

$$|\, \delta g \,| \ll |\, j \,|$$

$$|\, \delta W_p \,| \ll |\, U \,|$$

$$|\, \delta W_o \,| \ll |\, U \,|$$

so that they can be treated in a linear way. After inserting eq. (3-8) to eq. (3-10) into eq. (3-6), one obtains:

$$\delta H = \frac{\delta W_o - \delta W_p}{j(\underline{x}_p^o) + \alpha H^o} + \frac{U(\underline{x}_p^o) - U(\underline{x}_Q^o)}{[j(\underline{x}_p^o) + \alpha H^o]^2} \, \delta g_p \tag{3-11}$$

and the approximate value, H^o, is given by

$$H^o = \frac{U(\underline{x}_Q^o) - U(\underline{x}_p^o)}{j(\underline{x}_p^o) + \alpha H^o} \tag{3-12}$$

If H^0 is not sufficiently close to the actual value, the entire process must be iterated.

The next step in the derivation deals with δW_o and δW_p in eq. (3-11). We know that

$$W_p(\underline{x}_p) = U(\underline{x}_p) + T(\underline{x}_p) \tag{3-13}$$

$$W_o(\underline{x}_Q) = U(\underline{x}_Q) + T(\underline{x}_Q) \ . \tag{3-14}$$

Expressing these in a Taylor series:

$$W_p(\underline{x}_p) = U(\underline{x}_p^o) + [\mathrm{grad}^T U(\underline{x}_p^o)] \ \delta\underline{x}_p + T(\underline{x}_p^o) + O_2(T, \ \delta\underline{x}_p) \tag{3-15}$$

$$W_o(\underline{x}_Q) = U(\underline{x}_Q^o) + [\mathrm{grad}^T U(\underline{x}_Q^o)] \ \delta\underline{x}_Q + T(\underline{x}_Q^o) + O_2(T, \ \delta\underline{x}_Q) \ . \tag{3-16}$$

Neglecting second- and higher-order terms, one obtains for δW_p

$$\delta W_p = W_p(\underline{x}_p) - U(\underline{x}_p^o) = \underline{j}^T(\underline{x}_p^o) \quad \delta\underline{x}_p + T(\underline{x}_p^o) \tag{3-17}$$

$$\delta W_o = W_o(\underline{x}_Q) - U(\underline{x}_Q^o) = \underline{j}^T(\underline{x}_Q^o) \quad \delta\underline{x}_Q + T(\underline{x}_Q^o) \tag{3-18}$$

where

$$\left.\begin{aligned} \underline{j}(\underline{x}_P^o) &= \mathrm{grad}\ U(\underline{x}_p^o) \\ \underline{j}(\underline{x}_Q^o) &= \mathrm{grad}\ U(\underline{x}_Q^o). \end{aligned}\right. \tag{3-19}$$

However, $\underline{x}_p$ and $\underline{x}_Q$ are not independent of each other. Thus, we approximate $\underline{x}_Q$ on the geoid by

$$\underline{x}_Q \doteq \underline{x}_P + H\underline{n}_p \tag{3-20}$$

where

$$\underline{n}_p = - \begin{pmatrix} \cos\Phi_p & \cos\Lambda_p \\ \cos\Phi_p & \sin\Lambda_p \\ \sin\Phi_p & \end{pmatrix} \tag{3-21}$$

is the unit vector normal to the geopotential surface at P. Φ and Λ are respectively the astronomic latitude and longitude of the unit gravity vector at P. Relation (3-20) holds for the case that the curvature and the torsion of the plumb line are set to zero. Since we are only looking for an observation equation, this assumption is justified. Equation (3-20) has, however, to be linearized. Thus we have

$$\delta \underline{x}_Q = \delta \underline{x}_p + \delta H\, \underline{n}_p + H^o\, \delta \underline{n}_p \tag{3-22}$$

and

$$\Phi_p = \phi_p + \delta\Phi_p \tag{3-23}$$

$$\Lambda_p = \lambda_p + \delta\Lambda_p \tag{3-24}$$

$$\underline{n}_p = \underline{n}_p^o + \delta \underline{n}_p \tag{3-25}$$

and

$$\underline{n}_p^o = - \begin{pmatrix} \cos\phi_p & \cos\lambda_p \\ \cos\phi_p & \sin\lambda_p \\ \sin\phi_p & \end{pmatrix} \tag{3-26}$$

$$\delta \underline{n}_p = \begin{pmatrix} \cos\phi_p & \sin\lambda_p & \delta\Lambda_p + \sin\phi_p & \cos\lambda_p & \delta\Phi_p \\ -\cos\phi_p & \cos\lambda_p & \delta\Lambda_p + \sin\phi_p & \sin\lambda_p & \delta\Phi_p \\ -\cos\phi_p & \delta\Phi_p & & & \end{pmatrix}. \tag{3-27}$$

ϕ_p and λ_p in eqs. (3-23) to (3-27) are the geodetic latitude and longitude at P . Expressions for $\delta\Phi_p$ and $\delta\Lambda_p$ can be found in Hein (1982a: eq. (2-22) f.).

After inserting $\delta\Phi_p$, $\delta\Lambda_p$ into eq. (3-27) and the resulting expression into eq. (3-22) a new value for (3-18) can be obtained. Substituting this equation together with eq. (3-17) and δg_p (Hein 1982a:eq. (2-33) f.) in eq. (3-11), and after some lengthy, but simple arithmetic rearrangements, we obtain the linear observation equation of orthometric heights in the integrated geodesy adjustment model.

$$\delta H = X\,\delta x_p + Y\,\delta y_p + Z\,\delta z_p$$

$$+ \tau_Q\, T_Q + \tau_P\, T_P \tag{3-28}$$

$$+ \rho\left(\frac{T_r}{j}\right)_P + \beta\left(\frac{T_b}{jr}\right)_P + \lambda\left(\frac{T_l}{jr\cos b}\right)_P .$$

The indices of T in the third line of eq. (3-28) indicate differentiation with respect to spherical coordinates (r,b,l).

Thus, δH is linearized with respect to (x,y,z) of the surface point P and the disturbing potential T at Q and P, as well as the gravity disturbance and the deflections of the vertical. The coordinate unknowns form the deterministic term in the general collocation model (2-6), whereas the disturbing potential and its derivates are part of the stochastic signal $\underline{t}$.

The coefficients X, Y, Z in eq. (3-28) are given by the expression a = {X,Y,Z}

$$a = q^{-1}\,[H^o \sin\phi_P\,(j_{xQ}\cos\lambda_P + j_{yQ}\sin\lambda_P - j_{zQ}\cos\phi_P)\;\; a_{1P}$$

$$+ H^o\cos\phi_P\,(j_{xQ}\sin\lambda_P - j_{yQ}\cos\lambda_P)\,a_{2P} \tag{3-29}$$

$$+ \left(\frac{U_P - U_Q}{j_P + \alpha H^o}\right) a_{3P} + j_{xQ} - j_{xP}]$$

where

$$a_{ip} = \{X_{ip}, Y_{ip}, Z_{ip}\},\; i = 1, 2, 3$$

$$q = j_P + \alpha H^o + \cos\phi_P\cos\lambda_P\, j_{xQ} + \cos\phi_P\sin\lambda_P\, j_{yQ} + \sin\phi_P\, j_{zQ} \tag{3-30}$$

$$X_{1P} = j^{-1}\quad [\sin\phi\cos\lambda,\ \sin\phi\sin\lambda,\ -\cos\phi]\ [U_{xx}\ U_{xy}\ U_{xz}]^T \tag{3-31}$$

$$Y_{1P} = j^{-1}\quad [\sin\phi\cos\lambda,\ \sin\phi\sin\lambda,\ -\cos\phi]\ [U_{yx}\ U_{yy}\ U_{yz}]^T \tag{3-32}$$

$$Z_{1P} = j^{-1}\quad [\sin\phi\cos\lambda,\ \sin\phi\sin\lambda,\ -\cos\phi]\ [U_{zx}\ U_{zy}\ U_{zz}]^T \tag{3-33}$$

$$X_{2P} = \frac{1}{j\cos\phi}\,[\sin\lambda,\ -\cos\lambda,\ 0]\ [U_{xx}\ U_{xy}\ U_{xz}]^T \tag{3-34}$$

$$Y_{2P} = \frac{1}{j \cos \phi} [\sin \lambda, -\cos \lambda, 0] [U_{yx}\ U_{yy}\ U_{yz}]^T \tag{3-35}$$

$$Z_{2P} = \frac{1}{j \cos \phi} [\sin \lambda\ \ -\cos \lambda, 0] [U_{zx}\ U_{zy}\ U_{zz}]^T \tag{3-36}$$

$$X_{3P} = [-\cos \phi \cos \lambda, -\cos \phi \sin \lambda, -\sin \phi] [U_{xx}\ U_{xy}\ U_{xz}]^T \tag{3-37}$$

$$Y_{3P} = [-\cos \phi \cos \lambda, -\cos \phi \sin \lambda, -\sin \phi] [U_{yx}\ U_{yy}\ U_{yz}]^T \tag{3-38}$$

$$Z_{3P} = [-\cos \phi \cos \lambda, -\cos \phi \sin \lambda, -\sin \phi] [U_{zx}\ U_{zy}\ U_{zz}]^T. \tag{3-39}$$

Note that $j_{xQ} = \left.\frac{\partial j}{\partial x}\right|_Q$, $j_{yP} = \left.\frac{\partial j}{\partial y}\right|_P$, $U_{xx} = \frac{\partial^2 U}{\partial x^2}$, etc.

The coefficients for T in eq. (3-28) are given by

$$\tau_Q = q^{-1} \tag{3-40}$$

$$\tau_P = -\tau_Q. \tag{3-41}$$

If at the point Q on the geoid, one has $W_o = U_Q$ then, the unknown T_Q in eq. (3-28) drops out.

The other coefficients ρ, β, λ in eq. (3-28) are defined by the expression

$$b^* = \{\rho, \beta, \lambda\} .$$

$$\begin{aligned} b^* = q^{-1} [& H^o \sin \phi_p \, (j_{xQ} \cos \lambda_p + j_{yQ} \sin \lambda_p - j_{zQ} \cos \phi_p) \, b^*_{1p} \\ & + H^o \cos \phi_p \, (j_{xQ} \sin \lambda_p - j_{yQ} \cos \lambda_p) \, b^*_{2p} \\ & + (\frac{U_p - U_Q}{j_p + \alpha H^o}) \, b^*_{3p}] \end{aligned} \tag{3-42}$$

where $b^*_{ip} = \{\rho_{ip}, \beta_{ip}, \lambda_{ip}\}$, $i = 1, 2, 3$

$$\rho_{1p} = \sin (\phi - b) \tag{3-43}$$

$$\beta_{1p} = \cos (\phi - b) \tag{3-44}$$

$$\lambda_{1p} = 0 \qquad \text{(for } 1 = \lambda) \tag{3-45}$$

$$\rho_{2p} = 0 \tag{3-46}$$

$$\beta_{2p} = 0 \tag{3-47}$$

$$\lambda_{2p} = -1 \;/\; \cos\phi \tag{3-48}$$

$$\rho_{3p} = -j \cos(\phi - b) \tag{3-49}$$

$$\beta_{3p} = -j \sin(\phi - b) \tag{3-50}$$

$$\lambda_{3p} = 0 \qquad \text{(for } 1 = \lambda) \;. \tag{3-51}$$

Discussion of the observation equation

(1) Assume that orthometric heights are available in a digital terrain data base. Then we are able to express an observation equation of the type in eq.(3-28) for every height. If we hold the position fixed and, consequently, neglect the deterministic coordinate part in (3-28), the result is a linear system of equations that can be considered as a discrete analogue of the integral determination of the gravity potential (and its functionals) by topographic masses, see e.g. Reinhart (1968) - at least theoretically. The problem is the signal-to-noise ratio of H or δH in eq. (3-6), respectively, and corresponding unknowns. Both terms on the right hand side of eq. (3-6) are of the order of centimeters or decimeters.

(2) If the heights are on a grid, the covariance matrix $\underline{K}_{tt}$ has a regular structure, which can be used to compute its inverse analytically.

(3) Since eq. (3-28) is an observation equation of the collocation type, it enables us to combine it with all other gravity-related observations in one approach for the determination of the gravity potential and its functionals. Gravity gaps can be filled with heights as (pseudo-) observations, if the appropriate accuracy is available.

(4) It is obvious that an orthometric height cannot determine the horizontal position of a station. As a conclusion one might think that a development of H as a function of 3D coordinates is inappropriate. But the reason behind that is the ability to combine it with 3D baseline components derived by GPS or very long baseline interferometry (VLBI) as shown in the next section.

(5) If other than Helmert heights are available, the derivation can easily be changed by substituting a different denominator in eq. (3-2). In the case of normal orthometric heights, for example, g in eq. (3-2) has only to be replaced by normal gravity j and the linearization of eq. (3-8) is superfluous.

4. THE APPROACH FOR DETERMINING ORTHOMETRIC HEIGHTS USING GPS OBSERVATIONS

Present GPS receivers (MACROMETER™[2] and Texas Instruments TI 4100) use the phase observables of the Global Positioning System for determining relative base line components. Two treatments of the GPS observations in the integrated model are possible. The first one uses the phase as a direct observable and also considers the orbit to be unknown in the observation equation (Eißfeller and Hein 1984). Consequently, orbit integration is part of the solution.

For the second possibility, certain preprocessing provides Cartesian coordinate differences, $\underline{\Delta x}_{ij}^{obs} = (\underline{x}_j - \underline{x}_i)^{obs}$ as GPS (pseudo-) observations associated with a covariance matrix, $\underline{C}_{nn} = \underline{C}_{\Delta x \Delta x}$ in a "nearly conventional terrestrial" system. The basic relationship to those Cartesian coordinate differences, $\underline{\Delta x}$, is then

$$\underline{\Delta x}_{ij} = (1 + k)\, \underline{R}_{\omega}\, \underline{\Delta x}_{ij}^{obs} \quad , \tag{4-1}$$

where k is a small scale change and $\underline{R}_{\omega}$ is a rotation matrix for small rotations ω_x, ω_y, ω_z,

$$\underline{R}_{\omega} \doteq \begin{pmatrix} 1 & \omega_z & -\omega_y \\ -\omega_z & 1 & \omega_x \\ \omega_y & -\omega_x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix} = \underline{I} + \delta\underline{R}_{\omega} . \tag{4-2}$$

$\underline{I}$ is the identity matrix. Considering further, that

$$\underline{x} - \underline{x}^{obs} = \delta\underline{R}_{\omega}\, \underline{x} = \delta\underline{R}_{x}\underline{\omega} \tag{4-3}$$

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[2]MACROMETER is a trademark of Aero Service Division, Western Geophysical Company of America.

where

$$\delta \underline{R}_x = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \tag{4-4}$$

and introducing approximate coordinates $\underline{x}_i^o$, $\underline{x}_j^o$ the resulting linear observation equation is (Steeves, 1984)

$$\Delta \underline{x}_{ij}^{obs} - \Delta \underline{x}_{ij}^{o} = \delta \underline{x}_j - \delta \underline{x}_i - (\delta \underline{R}_x)_{ij} \, \delta \underline{\omega} - \delta k \, \Delta \underline{x}_{ij} \tag{4-5}$$

where $\Delta \underline{x}_{ij}^{o}$ are the differences between corresponding approximate coordinates, $\underline{x}_j^o$, $\underline{x}_i^o$. $\delta \underline{x}_j$ and $\delta \underline{x}_i$ are defined according to eq. (2-3).

Further,

$$(\delta \underline{R}_x)_{ij} = \begin{pmatrix} 0 & -\Delta z_{ij} & \Delta y_{ij} \\ \Delta z_{ij} & 0 & -\Delta x_{ij} \\ -\Delta y_{ij} & \Delta x_{ij} & 0 \end{pmatrix} \tag{4-6}$$

and

$$\delta \underline{\omega} = [\; \delta\omega_x \quad \delta\omega_y \quad \delta\omega_z \;]^T . \tag{4-7}$$

Thus, $\delta \underline{x}_j$, $\delta \underline{x}_i$, $\delta \underline{\omega}$ and δk form the deterministic unknowns ($\rightarrow \delta \underline{x}$) in eq. (2-6). There is no signal $\underline{t}$ in eq. (4-5).

Now, a linear system of equations of the type in eq. (2-6) can be set up for

- GPS base line components, $\Delta \underline{x}_{ij}$, see eq. (4-5) ;
- orthometric heights, see eq. (3-28) ;

and, furthermore, if available

- relative and absolute gravity observations, Δg, g;
- geopotential differences, ΔW_{ij};
- astronomical latitudes and longitudes, Φ , Λ;
- as well as any other terrestrial measurements.

The linear observation equations for the measurements above are given in Hein (1982a).

The corresponding combined solution yields geocentric coordinates, $\underline{x}$, as well as the signal, $\underline{t}$; the last consisting of the disturbing potential, T, and its first-order derivatives (gravity disturbance and components of the deflections of the vertical). By simple transformations one readily obtains the geoid heights, $N = T/j$, and the adjusted ellipsoidal heights or differences, respectively, so that the orthometric height difference can be determined by (1-1).

The necessary covariance matrix, $\underline{K}_{tt}$, can be determined from (global) covariance models using the characteristics of the local gravity field.

If the matrix $\underline{C} = \underline{C}_{nn} + \underline{R}\ \underline{K}_{tt}\ \underline{R}^T$ is positive definite and the observation stations under consideration are not all on a spheropotential surface, U = constant, then no datum defect appears in the integrated geodesy adjustment, see Hein and Landau (1983). If the initial approximate geocentric coordinates are not sufficiently accurate, the process must be iterated. As long as we are interested only in relative quantities, for example,orthometric height differences, we can introduce into the adjustment one geocentric position as fixed. A high accuracy for this position is not required.

Since the integrated geodesy adjustment is solved by a general collocation-type model, it has the capability to interpolate any functional of the gravity disturbing potential (geoid heights, deflections of the vertical, gravity gradients, etc.) in the area under consideration.

5. CONCLUSIONS

The derived integrated geodesy adjustment model offers a way of computing orthometric height differences as well as precise geoid heights taking advantage of observations of the satellites of the Global Positioning System or VLBI and using all other available terrestrial observations. In particular, it does not require that either orthometric heights or geoid heights, respectively, be considered error-free as assumed in traditional approaches. The combination has become possible by developing linear (pseudo-) observation equations for an orthometric height and the GPS base line components in the integrated model.

The underlying collocation-type model allows one to predict the geoid height and all other functionals of the gravity disturbing potential in the area under consideration. The consideration of noise in the data and the resulting error statistics provide a tool for a real assessment of the computed quantities.

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