

Energy Criterion for Fatigue Strength of Wood Structural Members

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This report describes a mathematical model for fatigue strength of cellulosic materials under sinusoidal loading. The model is based on the Reiner-Weissenberg thermodynamic theory of strength in conjunction with a nonlinear Eyring's three-element model. This theory states that failure depends on a maximum value of the intrinsic free energy that can be stored elastically in a volume element of the material. The three-element mechanical model, which consists of a linear spring in series with a parallel array of another linear spring and an Eyring dashpot, provides a good description of rheological material properties. The strength model system was able to predict rupture occurrence of polymers and wood structural members under constant and ramp loading with satisfactory results. For sinusoidal loading, the present study shows that the strength model system can predict time at fracture as a function of applied mean stress, amplitude of cyclic stress, and stress frequency. Numerical examples with model parameters evaluated for small Douglas-fir beams are presented.

Introduction

Wood structural members exhibit time-dependent rupture behavior. The design of these members requires an appropriate consideration of the effect of load duration. The estimation of load-duration effects for specified loading conditions is usually obtained from empirical mathematical models whose parameters are determined from creep rupture test data. In the present study, we developed a mathematical model for time-dependent strength of wood structural members under sinusoidal load with low heat generation due to cyclic stressing. The analysis is based on the Reiner-Weissenberg energy criterion and is an extension of a recent paper that considers the viscoelastic effects due to constant and ramp loading on strength of wood (Liu and Schaffer, 1995).

The effects of sinusoidal loading on time-dependent strength of polymeric filaments were analyzed by Coleman (1956) and Coleman and Knox (1957), and on wood by Liu et al. (1994). Their mathematical models were based on the theory of reaction rate for fracture of solids proposed by Tobolsky and Eyring (1943). Common features of their models include (a) models for sinusoidal load and constant load are identical in form and (b) fatigue life is independent of stress frequency. The model in the present study also has the same features when used on wood, but stress frequency is contained in the mathematical formulation.

The Reiner-Weissenberg thermodynamic theory of strength (Reiner, 1964) was first applied to the study of time-dependent fracture of wood under various loading conditions by Bach (1973). The theory states that failure depends on a maximum value of the intrinsic free energy that can be stored elastically in a volume element of the material. In conjunction with the strength theory, Bach used a linear mechanical model consisting of a series arrangement of a spring, a Kelvin chain, and a dashpot to represent the material rheological properties. He also

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modified the theory by neglecting the instantaneously stored energy so that stress relaxation rupture would be possible. No quantitative or numerical verification of the model was attempted.

A critical strain-energy-density failure criterion with a linear, viscoelastic, four-element Burger model was reported by Fridley et al. (1992). Their approach was similar to Bach's in that both used an energy failure criterion in conjunction with a linear rheological mechanical model.

With the linear mechanical model proposed by Bach (1973) replaced by a three-element model consisting of a spring in series with a parallel array of another spring and an Eyring dashpot (Fig. 1) (Reichardt et al., 1946), the Reiner-Weissenberg theory has successfully predicted the creep rupture time of polymers (Cherry and Teoh, 1983; Teoh, 1990; Teoh et al., 1992), tropical wood polymer composites (Teoh et al., 1987; Boey and Teoh, 1990), and wood (Liu and Schaffer, 1995). The model (Fig. 1) is an alternative to Eyring's three-element model, which was considered to be the most successful when it was introduced (Morton and Hearle, 1962). In the case of creep rupture, the strength model system has the special features to predict (a) the upper stress limit at which the material ruptures immediately upon application of load and (b) the lower stress limit, or threshold stress, at which the material can sustain the applied load indefinitely. In the case of rupture due to sinusoidal load, these special features also define the allowable magnitudes of the mean stress, the amplitude of the cyclic stress, and stress frequency.

Failure Condition and Theological Model

Reiner and Weissenberg (Reiner, 1964) postulate that failure of a viscoelastic material with negligible volumetric changes depends on a maximum value of the intrinsic free energy that can be stored elastically in the volume element of the material. The intrinsic free energy may be called the strain work WC , "strain" denoting the recoverable part of deformation. Failure will occur at the time t_f when

$$w_c = \int_0^{t_f} (\dot{w} - \dot{D}) dt = R_1 \quad (1)$$

where the over-dot ($\dot{\quad}$) denotes derivative with respect to time

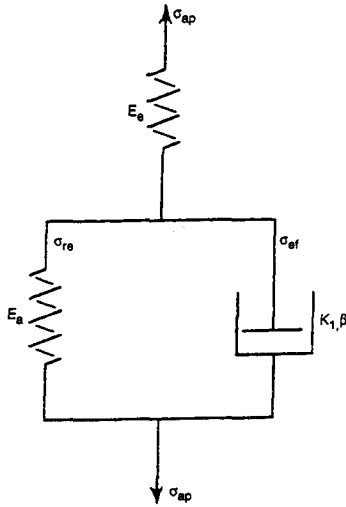


Fig. 1 An alternative to Eyring's three-element mechanical model (Reichart et al., 1946)

t , \dot{w} is rate of stress-work done on the material, \dot{D} is rate of dissipation of nonelastic energy, and R_1 is a material constant, which may be called the resilience of the material. Equation (1) may be called the failure condition.

In conjunction with the failure condition, a rheological model needs to be developed describing the mechanical behavior of the material under load. Such a model consists of a linear spring in series with a parallel array of another linear spring and an Eyring dashpot (Fig. 1). The latter linear spring is anelastic because the Eyring dashpot as described in Eq. (5) is an element describing the activated rate process of plastic deformation (Fotheringham and Cherry, 1978a, b). The strain work in Eq. (1) will be obtained directly from the strain energy stored in the two linear springs in Fig. 1. The model has been successfully applied to predict creep rupture of polymers, wood polymer composites, and wood, as mentioned previously.

Applied Stresses

Consider a sinusoidal stress applied to the mechanical model in Fig. 1,

$$\sigma_{ap} = \sigma_c + \sigma_0 \sin(\omega t + \phi) \quad (2)$$

where σ_c is constant mean stress, σ_0 is amplitude of cyclic stress, ω is circular frequency, t is time, and ϕ is phase angle. The quantity $\tan \phi$ is the loss factor and is another way of characterizing the viscous effect (Morton and Hearle, 1962).

It is convenient to analyze the mechanical responses associated with the constant and cyclic stress components separately.

(a) Constant Stress, σ_c . With $\sigma_{ap} \equiv \sigma_c$, we can define the recovery stress acting on the anelastic spring as $\sigma_{re,c}$ and the effective stress acting on the Eyring dashpot as $\sigma_{ef,c}$. The following strains are likewise defined

$$\epsilon_{e,c} = \sigma_c / E_e \quad (3)$$

and

$$\epsilon_{a,c} = \sigma_{re,c} / E_a \quad (4)$$

where E_e and E_a are the moduli of the elastic and anelastic springs, respectively. The strain rate of the Eyring dashpot is equal to the strain rate of the anelastic spring:

$$\dot{\epsilon}_{a,c} = K_1 \sinh(\beta_c \sigma_{ef,c}) \quad (5)$$

where K_1 is a function of the activation energy and β_c is the

stress coefficient, which may vary with the type of loading (Liu et al., 1994).

The strain rate in Eq. (5) can also be obtained from Eq. (4) as

$$\dot{\epsilon}_{a,c} = \dot{\sigma}_{re,c} / E_a = (\dot{\sigma}_c - \dot{\sigma}_{ef,c}) / E_a \quad (6)$$

Since $\dot{\sigma}_c = 0$, we obtain from Eqs. (5) and (6)

$$-E_a K_1 \int_0^{t_f} dt = \int_{\sigma_c}^{\sigma_{ef,c}} \frac{d\sigma_{ef}}{\sinh(\beta_c \sigma_{ef})} \quad (7)$$

where t_f is time and σ_c and $\sigma_{ef,c}$ are stresses at failure. Time to failure $t_f = 0$ when $\sigma_c = \sigma_{ef,c}$, that is, the anelastic spring is not taking any load. For the evaluation of the integrals in Eq. (7), $\sigma_{ef,c}$ needs to be determined first.

(b) Cyclic stress, $\sigma_0 \sin(\omega t + \phi)$. With $\sigma_{ap} \equiv \sigma_0 \sin(\omega t + \phi)$, the stresses in Fig. 1 and their corresponding strains are denoted with a subscript replacing the subscript c, indicating an association with the constant stress in the related parameters. Thus, we also define

$$\epsilon_{a,s} = \epsilon_0 \sin \omega t \quad (8)$$

where ϵ_0 is the amplitude of the strain and

$$\dot{\epsilon}_{a,s} = K_1 \sinh(\beta_s \sigma_{ef,s}) \quad (9)$$

For low heat generation or plastic energy dissipation, consider the linear, low stress region of the hyperbolic sine viscosity relation in Eq. (9) (Krausz and Eyring, 1975). We therefore obtain from stress equilibrium of the mechanical model

$$\begin{aligned} \sigma_0 \sin(\omega t + \phi) &= E_a \epsilon_0 \sin \omega t + \epsilon_0 \omega \cos \omega t / K_1 \beta_s \\ &= \epsilon_0 \lambda (E_a \sin \omega t / \lambda + \omega \cos \omega t / K_1 \beta_s \lambda) \\ &= \epsilon_0 \lambda (\sin \omega t \cos \phi + \cos \omega t \sin \phi) \\ &= \epsilon_0 \lambda \sin(\omega t + \phi) \end{aligned} \quad (10)$$

where

$$\lambda = \left[E_a^2 + \left(\frac{\omega}{K_1 \beta_s} \right)^2 \right]^{1/2} \quad (11)$$

and

$$\tan \phi = \frac{\omega}{K_1 \beta_s E_a} \quad (12)$$

Equation (10) shows

$$\sigma_0 = \epsilon_0 \lambda \quad (13)$$

We note $\tan \phi$ in Eq. (12) contains the circular frequency ω , and σ_0 and ϵ_0 are related by Eq. (13).

Strain Energy

Strain energy stored elastically in a volume element of the material can be calculated by

$$Q = \int \sigma \dot{\epsilon} dt \quad (14)$$

in which the limits of integration depend on the form of the strain rate. Evaluation of the strain energy for the two linear springs can again be made separately using the mechanical responses associated with the stress components described previously.

(a) Elastic Spring. The stress acting on the elastic spring can be written as

$$\sigma_e = \sigma_c \Delta(t) + \sigma_0 \sin(\omega t + \phi) \quad (15)$$

and the corresponding strain as

$$\epsilon_c = \sigma_c / E_c \quad (16)$$

In Equation (15), for ease of integral calculation in Eq. (14), we have introduced the unit step function $\Delta(t)$ with the properties (Flügge, 1967)

$$\Delta(t) = 0 \quad \text{for } t < 0$$

$$\Delta(t) = 1 \quad \text{for } t > 0$$

and

$$\dot{\Delta}(t) = \delta(t)$$

where $\delta(t)$ is the Dirac delta function with the properties

$$\delta(t) = 0 \quad \text{for } t \neq 0$$

$$\delta(t) = \infty \quad \text{for } t = 0$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

Substituting Eqs. (15) and (16) into Eq. (14) and integrating from $\omega t = 0$ to 2π when the integrand is a function of ω . (We note in a complete cycle with ωt increasing from 0 to 2π , the elastically stored strain energy is 0; when the number of cycles is large, we need to consider whole cycles only.) or from 0 to the upper limit of strain when the integrand is independent of ω , we obtain the strain energy

$$Q_c = \frac{\sigma_c^2}{E_c} + \frac{\sigma_c \sigma_0}{E_c} \sin \phi \quad (17)$$

(b) Anelastic Spring. The stress acting on the anelastic spring can be written as

$$\sigma_a = \sigma_{re,c} + E_a \epsilon_0 \sin \omega t \quad (18)$$

and the corresponding strain as

$$\epsilon_a = \epsilon_{a,c} + \epsilon_0 \sin \omega t \quad (19)$$

The strain energy obtained likewise from Eq. (14) is

$$Q_a = E_a \epsilon_{a,c}^2 / 2 \quad (20)$$

Fatigue Life

From Eqs. (1), (17), and (20)

$$w_c = Q_c + Q_a = R_1 \quad (21)$$

which leads to

$$\epsilon_{a,c} = \left[\frac{2}{E_a} \left(R_1 - \frac{\sigma_c^2}{E_c} - \frac{\sigma_c \sigma_0}{E_c} \sin \phi \right) \right]^{1/2} \quad (22)$$

Since

$$\bar{\sigma}_{ef,c} = \bar{\sigma}_c - E_a \epsilon_{a,c} \quad (23)$$

fatigue failure time can now be obtained from Eq. (7) as

$$t_f = \left(\frac{1}{E_a K_1 \beta_c} \right) \ln \left[\frac{\tanh \left(\frac{\beta_c \bar{\sigma}_c}{2} \right)}{\tanh \left(\frac{\beta_c \bar{\sigma}_{ef,c}}{2} \right)} \right] \quad (24)$$

We note when $\bar{\sigma}_c = \bar{\sigma}_{ef,c}$, $t_f = 0$ in Eq. (24) and, from Eqs. (22) and (23), we obtain

$$\bar{\sigma}_c = -\frac{\bar{\sigma}_0}{2} \sin \phi + \frac{1}{2} (\bar{\sigma}_0^2 \sin^2 \phi + 4E_c R_1)^{1/2} \quad (25)$$

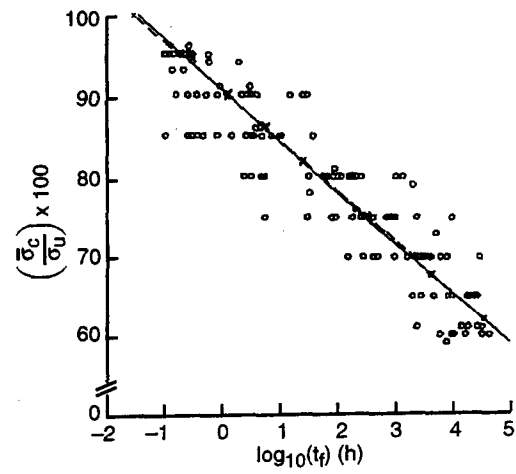


Fig. 2 Variation of constant stress level at failure with logarithm of time. (---, Eq. (24); —, Eq. (27); o Wood (1951); $\sigma_u = 53.1$ MPa, short-term strength.)

Also, when $\bar{\sigma}_{ef,c} = 0$, $t_f = \infty$ in Eq. (24) and we obtain

$$\bar{\sigma}_c = -\frac{E_a \bar{\sigma}_0 \sin \phi}{E_c + 2E_a} + \left[\left(\frac{E_a \bar{\sigma}_0 \sin \phi}{E_c + 2E_a} \right)^2 + \frac{2E_a E_c R_1}{E_c + 2E_a} \right]^{1/2} \quad (26)$$

Equations (22) to (26) agree with the results presented by Liu and Schaffer (1995) for the case of constant load where $\bar{\sigma}_0 = 0$. Equations (25) and (26) define the allowable magnitudes of $\bar{\sigma}_c$, $\bar{\sigma}_0$, and ω , which serve as a basis for design considerations.

Numerical Examples and Discussion

The test data on bending strength of small, clear Douglas-fir beams under constant load reported by Wood (1951) were used to evaluate the model parameters. Wood's data are represented by

$$(\bar{\sigma}_c / \sigma_u) \times 100 = 112.8 - 2.73 \ln(t_f) \quad (27)$$

where $\sigma_u = 53.1$ MPa, assumed as the ultimate or short-term strength of Douglas-fir, and t_f is in seconds.

For constant load, $\bar{\sigma}_0$ in Eq. (22) must be set equal to 0 before t_f is calculated using Eq. (24). The model parameters thus evaluated are (Liu and Schaffer, 1995)

$$\begin{aligned} E_c &= 11,200 \text{ MPa} \\ E_a &= 150 \text{ MPa} \\ R_1 &= 0.34 \text{ MJ m}^{-3} \\ \beta_c &= 0.5984 \text{ MPa}^{-1} \\ K_1 &= 8.8779 \times 10^{-17} \text{ s}^{-1} \end{aligned}$$

The numerical results are plotted in Fig. 2.

Because of the small magnitude of K_1 , the phase angle ϕ evaluated from Eq. (12) stays close to $\pi/2$ for practical values of ω and β_c . We therefore set $\phi = \pi/2$. To demonstrate the effect of variations of $\bar{\sigma}_c$ and $\bar{\sigma}_0$ on lifetime t_f , we also assign values for $\bar{\sigma}_c$ and $\bar{\sigma}_0$ in Eqs. (22) to (24) as follows:

$$\frac{\bar{\sigma}_c}{\sigma_u} = \begin{cases} 0.6 \\ 0.65 \\ 0.7 \end{cases} \quad \frac{\bar{\sigma}_0}{\sigma_u} = \begin{cases} 0.065 \\ 0.13 \\ 0.26 \end{cases} \quad (28)$$

With these input data, results from Eq. (24) are presented in

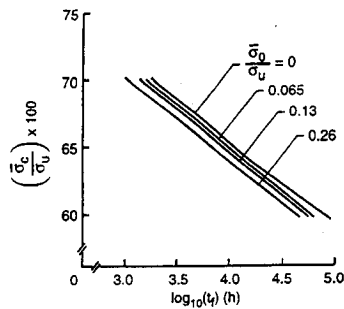


Fig. 3 Variation of constant stress level at failure with logarithm of time for several stress amplitude levels, $\sigma_u = 53.1$ MPa

Fig. 3. Clearly, the lifetime increases with a decreasing value of $\bar{\sigma}_c$ for any given value of $\bar{\sigma}_0$, and with a decreasing value of $\bar{\sigma}_0$ for any given value of $\bar{\sigma}_c$. The effect on lifetime of ω , though masked by the smallness of K_1 in Eq. (12), is equivalent to that of $\bar{\sigma}_0$, as can be seen from Eqs. (12) and (22). This also explains the uncertainties of the fatigue test data on wood with respect to frequency as reported by Kohara and Okuyama (1992).

In Liu et al. (1994), a parameter defining the fraction of unbroken bonds must be determined from cyclic fatigue tests to be used for fatigue-life prediction. In the present study, all model parameters can be evaluated from creep tests only. Indeed, the parameter β , in Eq. (9) needs also to be determined from cyclic fatigue tests, but its effect is obliterated when the phase angle stays at $\pi/2$.

Conclusions

We applied the Reiner-Weissenberg strength theory together with an Eyring's three-element mechanical model to predict the fatigue life of wood structural members under sinusoidal load. The Reiner-Weissenberg strength theory postulates that elastic energy stored viscoelastically prior to rupture is independent of the loading path. The parameters in the derived mathematical model are evaluated from a corresponding model for constant load, for which test data of Douglas-fir beams under constant bending load are available. The models for sinusoidal load and constant load are identical in form.

Our results indicate that fatigue life under cyclic bending load for a polymeric material such as Douglas-fir wood has the following characteristics:

1. For a given mean stress, fatigue life increases with a decrease in amplitude of cyclic stress.
2. For a given amplitude of cyclic stress, fatigue life increases with a decrease in mean stress.
3. Stress frequency has an effect on fatigue life through the phase angle only. For Douglas-fir wood and materials having similar properties, the phase angle stays at $\pi/2$.

Therefore, the effect of frequency on fatigue life for these materials is not significant.

Our mathematical model includes two special features: (a) relative magnitude of mean stress, amplitude of cyclic stress, phase angle, and hence stress frequency, are limited by Eq. (25), which defines the state of immediate rupture, and (b) threshold state corresponding to a fatigue life of infinity is defined by Eq. (26). These features are of considerable interest in development of building codes and assessing residual life.

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