THE WEST KENTUCKY GEOID PROJECT

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Abstract

The use of the Global Positioning System (GPS) for horizontal control surveys is now a routine part of survey practice. The use of GPS for leveling, however, has seen less widespread. Fundamental to the effective use of GPS for leveling is an understanding of both the local and regional geoid.

The West Kentucky Geoid Project was undertaken to map the ellipsoid-geoid separation in the eight county area of Kentucky west of the Tennessee River. This area covers some 2,100 square miles, and is roughly the size of the State of Delaware. In the first stage of this project, the GEOID99 model was used to prepare a regional isoline map and a three dimensional model of geoid separation. These allow identification of areas exhibiting both regular and irregular ellipsoid-geoid separation surfaces.

In the second stage of the project, the GEOID99 model again was used to prepare isoline maps, this time at the county level. These maps are intended to assist surveyors in planning GPS surveys, and in developing a methodology to improve the results of GPS leveling. The research reported in this paper includes the results of mapping and a discussion of how the maps can be used in GPS leveling.

Introduction

As noted by Kepler (1618),

The surface of the Earth is not perfectly round, but follows the shape of the oceans, if these were still. There is a natural reason for the roundness of the Earth: We see that the Earth and the oceans have an inner force which attracts other bodies and joins them together. This force is known as gravity.

Surveyors recognize this force whenever they determine an elevation by differential leveling. However, while use of the Global Positioning System (GPS) for horizontal control surveys is now a routine part of survey practice, the use of GPS for leveling has been less widespread. The principal reason for this appears to be the difficulty of relating ellipsoidal heights from GPS observations to orthometric elevations based on the North American Vertical Datum of 1988.

While the theoretical relationships between the ellipsoid and geoid have been the subject of much work, practical conversions of ellipsoidal and orthometric heights require care by the practicing surveyor. The distinction between ellipsoidal height, orthometric height, and the role of the GEOID99 model of the U.S. National Geodetic Survey (NGS) in height conversions are discussed below.

Ellipsoidal heights. The position of a point, P, on the Earth's surface can be converted from ellipsoidal to Cartesian coordinates, as obtained by GPS as

$$x_p = (N + h)\cos\phi\cos\lambda\ , \tag{1}$$

$$y_p = (N + h)\cos\phi\sin\lambda\ , \quad \text{and} \tag{2}$$

$$z_p = \left[\left(1 - e^2\right)N + h\right]\sin\phi\ , \quad \text{where} \tag{3}$$

ϕ = geodetic latitude,
λ = geodetic longitude,
N = normal to the ellipsoid,
e = eccentricity, and
h = height above the ellipsoid (Torge, 1980).

From basic geodesy,

$$e = \frac{\sqrt{a^2 - b^2}}{a}\ , \quad \text{and} \tag{4}$$

$$N = \frac{a}{\left(1 - e^2\sin^2\phi\right)^{1/2}}\ , \quad \text{where} \tag{5}$$

a = ellipsoid major semidiameter, and
b = minor semidiameter of the ellipsoid. (Torge, 1980).

Using the coordinates defined above, ellipsoidal height can be found from

$$h = \frac{\sqrt{x^2 + y^2}}{\cos\phi}\ , \quad \text{where} \tag{6}$$

$$\phi = \tan^{-1}\left[\frac{z}{\sqrt{x^2 + y^2}}\left(1 - e^2\frac{N}{N + h}\right)^{-1}\right] \quad \text{(Torge, 1980).} \tag{7}$$

It should be noted that equations (6) and (7) are solved by iteration.

Orthometric heights. While the above equations describe the height of the point, P, with respect to the mathematically defined ellipsoid, it is also possible to describe the height of point P by giving its *orthometric height* above the physically determined *geoid*. As defined by the NGS, the *geoid* is the *equipotential surface of the Earth's gravity field which best fits, in a least squares sense, global mean sea level (NGS, 2000). Orthometric height* may be defined as the distance between a specific survey station and the reference datum (normally the geoid) as measured along the plumb line (Wolf & Ghilani, 2002). To simplify the discussion below, the shape of the Earth is assumed to be a sphere. A more rigorous treatment of this topic can be found in Torge (1980).

In reflecting on the above definition of the geoid, two things must be noted. First, mean sea level is a very difficult surface to define accurately. This is due to the influence of coastal irregularities, the 18.5 year metonic cycle, and global sea level changes. Second, because the geoid is defined by the Earth's gravity field, the surface of the geoid is irregular. This is because the force of gravity is the resultant of the gravitational attraction of the Earth (which varies with the Earth's mass distribution), and the angular velocity of the Earth's rotation. Mathematically, the gravitational attraction *F* on an object of mass *m* located at sea level is found from

$$F = m\,a\,, \quad \text{where} \tag{8}$$

$$a = \frac{G\,M_e}{R_e^2}\,, \quad \text{and where} \tag{9}$$

m = object mass (gm),
a = acceleration (cm/s^2),
G = universal gravitational constant ($6.672 \times 10^{-8}\ cm^3\ g^{-1}\ s^{-2}$),
M_e = mass of the Earth (5.98×10^{27} g) and
R_e = non-critical radius of the Earth (6.371×10^8 cm) (Serway, 1992).

The centrifugal force *f* of the Earth's rotation can be found from

$$f = w^2\, R_e \cos f\,, \quad \text{where} \tag{10}$$

$$w = 7{,}292{,}115x10^{-11}\, rad / \sec \quad \text{and where} \tag{11}$$

ω = angular velocity,
ϕ = latitude, and
86, 164 = number of seconds in a sidereal day (Tsuboi, 1983).

The centrifugal force *f* at latitude ϕ acts in a direction that is parallel to the equator and perpendicular to the Earth's axis of rotation. The component of this force that acts radially at the surface is $f \cos f$, from which it follows that

$$f\,\cos f = w^2\, R_e \cos^2 f\,. \tag{12}$$

The resultant of a vector solution for equations (9) and (12) is the force of gravity, *g*, which may be approximated by

$$g = \frac{G\,M e}{R e^2} - w^2\, Re \cos^2 f\,. \quad \text{(Tsuboi, 1984)} \tag{13}$$

While equation (13) assumes spherical mass distribution, actual mass distribution within the Earth varies, thus influencing gravitational attraction at a specific point. The result, of course, is that the geoid assumes an irregular surface. This surface can be mapped based on field measurements of gravity, and the separation between the ellipsoid and the geoid is related through

$$H = h - N \,, \quad \text{where} \tag{14}$$

H = orthometric height (above geoid),
h = height above the ellipsoid, and
N = ellipsoid-geoid separation.

While the discussion above presents basic relationships, the development of a mathematical model of the geoid requires much more rigorous analyses. In addition, if the ellipsoid-geoid separation is to be accurately determined, it is necessary to have both a dense network of gravity measurements and an accurate digital terrain model (DTM). These topics are beyond the scope of this paper. However, an examination of geopotential models is provided by Smith and Milbert (2000), by Milbert and Schultz (1993), and by Milbert (1995 & 1991). Additional work is reported by Rapp (1996 & 1992), and by Rapp and Nerem (1994). A discussion of the use of digital terrain modeling data for determining ellipsoid-geoid separation is provided by Smith and Roman (1999). Finally, conversion of GPS observations to geoid height is discussed by Milbert and Smith (1996) and by Hein (1985).

GEOID99. From the discussion above, it follows that if the value of N can be determined, orthometric elevation and ellipsoidal height can be related. The GEOID99 model, as developed by the NGS, provides the user with the ellipsoid-geoid separation, N, as a function of geodetic position. As described by the NGS, GEOID99 is a hybrid mathematical model that is based on the G99SSS geoid model (NGS, 2000). GEOID99 supplements the gravimetric geoid in G99SSS with NAD83 GPS heights on some 6,169 NAVD88 benchmarks (NGS,2000). According to the NGS (2000), GEOID99 undulations have a 4.6 cm RMS difference when compared to GPS heights on NAVD88 benchmarks.

Procedure

Based on the above discussion, the West Kentucky Geoid Project was developed to prepare regional and county maps of western Kentucky showing the ellipsoid-geoid separation as determined from the GEOID99 model. It was intended that these maps would assist the practicing surveyor in relating GPS and orthometric heights.

Regional map. The function of the regional map is to show general trends in geoid separation. While in some areas the geoid assumes a constant slope, in other areas doming or depression in the geoid results due to local gravity anomalies. The GEOID99 model was used to determine ellipsoid-geoid separation at 1 minute intervals of latitude and longitude from 36E25'N 88E00'W to 37E20'N 89E40' W. The geodetic positions used for determining the ellipsoid-geoid separation from GEOID99 were converted to state plane coordinates for the Kentucky South Zone (KYSPCS 83/94) using the CORPSCON program developed jointly by NGS and the U.S. Army Corps of Engineers. The result of this work was a file of 5,600 sets of three dimensional coordinates defining the ellipsoid-geoid separation.

Both an isoline map and a perspective model of ellipsoid-geoid separation were developed for the eight county region of western Kentucky using the SURFER (version 7) software package (Golden Software, 1999). Both the map and model were prepared using a Delaunay triangulation with linear interpolation. The resulting grid was then smoothed using matrix smoothing.

The regional ellipsoid-geoid separation isoline map was plotted at a 0.1 m isoline. As shown in figure 1, the isoline map was registered onto a map of state and county boundaries digitized from a 1:250,000 U.S. Geological Survey quad. Stations for which both geoid and ellipsoid heights have been determined are plotted as well. The model of the separation surface is shown in figure 2. The surface is presented as a inclined orthogonal model drawn as viewed from the southwest. An isoline map of the model surface is superimposed.

The accuracy of figure 1 was tested by comparing the separation shown on the map to actual separation measurements made by the National Geodetic Survey. Unfortunately, only six stations in the Jackson Purchase have dual heights. However, the errors obtained are summarized in table 1. This shows errors to range from 0.04 m to –0.06 m, based on separation values interpolated from figure 1.

Station	Map Separation (m)	Known Separation (m)	Error (m)
V22	-28.70	-28.74	0.04
X128	-28.82	-28.80	-0.02
R393	-28.17	-28.11	-0.06
KEVIL	-28.32	-28.30	-0.02
E390	-28.75	-28.72	-0.03
J20	-28.32	-28.29	-0.03

Table 1. Comparison of separation as shown on map (figure 1) with that observed in field.

County maps. The regional map discussed above was supplemented by larger scale separation maps. Figure 3, which includes a three county area of western Kentucky, was developed to assist surveyors working along the levee system of the Mississippi River. This map is similar to figure 1, but at larger scale. An isoline interval of 0.01 m is employed. In addition, individual county maps, similar to figure 3, were prepared as part of this project.

Finally, figure 5 shows the geoid separation in McCracken County superimposed on a digital terrain model of the county. The intent of this maps was to investigate any obvious correlation between topography and the ellipsoid-geoid separation.

Application in field surveys

Various approaches have been used or suggested for employing GPS surveys for the densification of vertical control. The technique developed by the NGS is discussed by Zilkoski (1990), and the procedures discussed therein have been shown to provide centimeter level uncertainties over a 50 kilometer distance. The technique proposed requires high-accuracy field procedures, careful network design, and rigorous scrutiny of field work results if the accuracy cited is to be achieved.

As noted by Zilkoski (1990), the most significant source of error in vertical control densification by GPS is the geoid height difference estimate. As demonstrated in the research reported herein, the slope of the geoid can change radically between widely spaced benchmarks. Consequently, it is necessary to employ a dense network of known benchmarks referenced to a common datum within the GPS leveling network in order to define geoid slope. It must follow, as well, that any systematic errors in the orthometric leveling employed to determine benchmark elevation must be corrected in order to avoid significant errors in GPS-derived orthometric heights.

In particular, the surveyor must ensure that all benchmarks used in the network are stable and have not been subject to movement since the orthometric height was initially obtained. Check leveling should be performed in areas where there is any question of stability. Further, it should also be apparent that elevations referred to the North American Vertical Datum of 1988 (NAVD88) are to be preferred to a previously defined datum. This is because earlier leveling may have been improperly constrained during adjustment.

Based on studies conducted by NGS in Boulder County, Colorado, and Summit County, Ohio, Zilkoski (1990) recommends a series of steps to be employed in determining accurate GPS-derived orthometric heights. The steps suggested by Zilkoski (1990) may be summarized as follows:

1. The geoid in the survey area should be analyzed during the planning stage in order to define geoid slope. The analysis should identify any additional leveling or gravity observations needed to describe the slope of the geoid.
2. A detailed and thorough study of existing leveling is necessary. This part of the work should show which

benchmarks can be occupied by GPS equipment and which benchmarks can be considered as stable.

3. A three dimensional minimum constraint least squares solution is necessary to compare

   GPS-derived coordinate with the results of higher order control. The comparison should indicate where the higher order survey can be used to constrain lower order work.
3. With the above steps completed, it is possible to compare GPS elevations and orthometric heights.
4. Examination of data for outliers is the next step. Naturally, the outliers should be removed from the data set.
5. The local geoid now can be analyzed in detail, and the slope of the geoid estimated. Differences between ellipsoidal heights from GPS and orthometric heights from leveling can be obtained from
6. Vincenty's method (1987) next can be used to solve for geoidal slope and scale.
7. A comparison should be made between adjusted GPS orthometric height differences and leveling-based orthometric height differences. This comparison should indicate whether additional parameters are needed to define changes in geoid slope within the project area.
8. Solve for GPS—derived heights using a three-dimensional least squares adjustment constrained to the height values of existing benchmarks.
9. Estimate the accuracy of the GPS-derived orthometric heights by using the data obtained in steps 3-9 above.

Conclusions

The above project was undertaken in order to provide the practicing surveyors in a specific area with basic information on the use of GPS for determining elevations. Several things were obvious from this study. First, it is apparent from the figures presented that the separation, N, between the ellipsoid and the geoid isn't constant, even over a fairly limited area. Further, even where the ellipsoid-geoid separation slope is constant, the slope surface may well be inclined over the area of interest. In either case, this precludes the simple calculation of an average value for N based on field observations.

Second, correction of GPS observations to the geoid is possible using GEOID99. However, the elevation resulting can be no better than the resolution of the GEOID99 model. Whether the accuracy resulting is sufficient for the purpose of the GPS measurements involved is a professional judgment.

Third, accurate determination of geoid slope is central to determination of GPS-derived orthometric heights. The rigorous procedure suggested by Zilkoski (1990) as described above is suggested for this purpose, and as noted by Zilkoski, Carlson, and Smith (2001) the latest version of GEOID99 has lessened this problem.

Finally, there remains much work to be done by the surveying profession in developing an understanding of the geometric and mathematical relationships inherent in GPS surveying. Intelligent and productive use of GPS equipment must be based on a thorough understanding of the basic principles and limitations governing GPS operation

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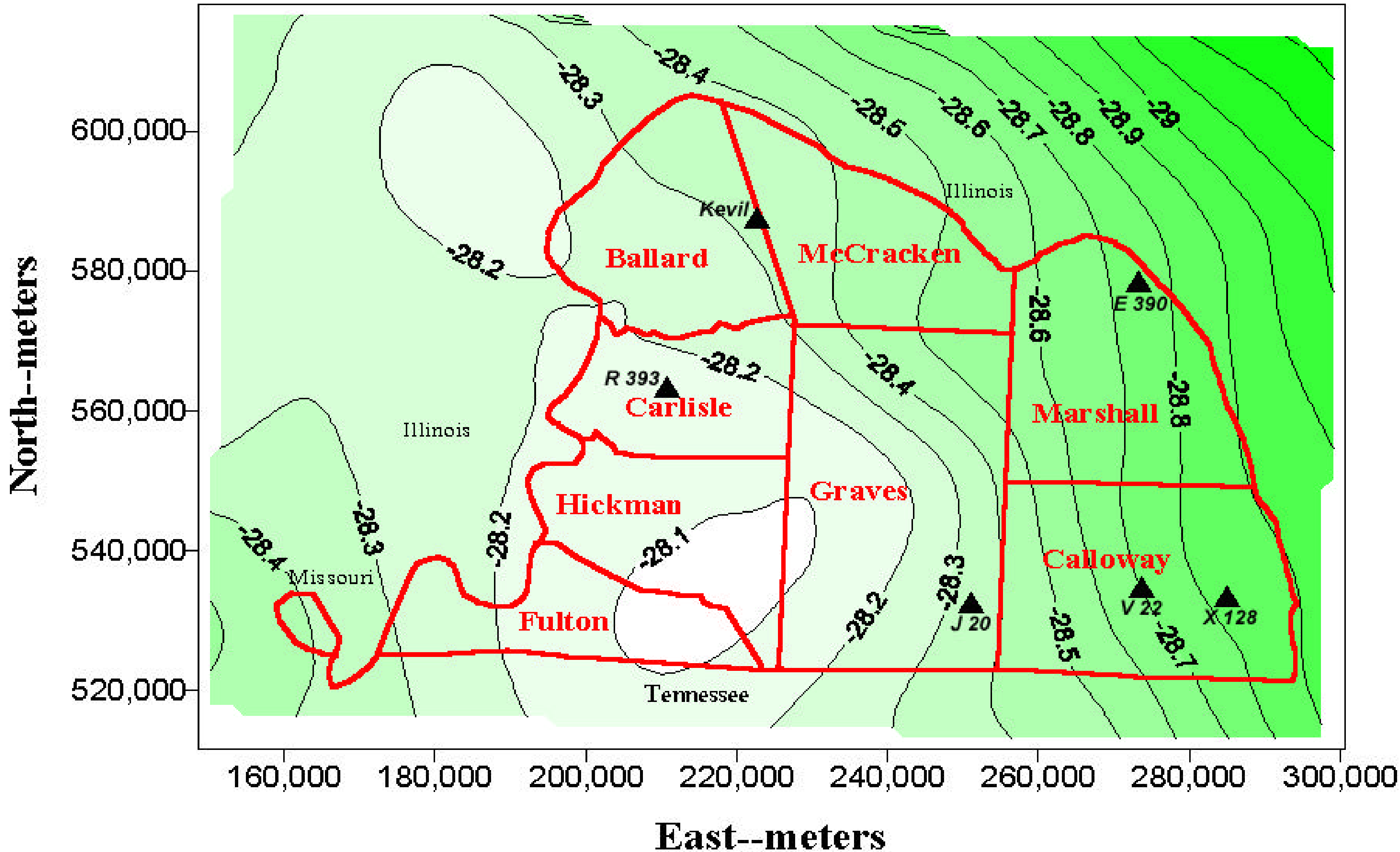

Notes:

1. Kentucky county names and boundaries are shown in red.
2. Contours denote ellipsoid-geoid separation in meters.
3. Grid coordinates are for the Kentucky State Plane Coordinate System, South Zone, in meters.
4. Triangles denote points for which dual heights (ellipsoid and geoid) exist.

Figure 1. Ellipsoid-geoid separation (meters), Jackson Purchase, Kentucky

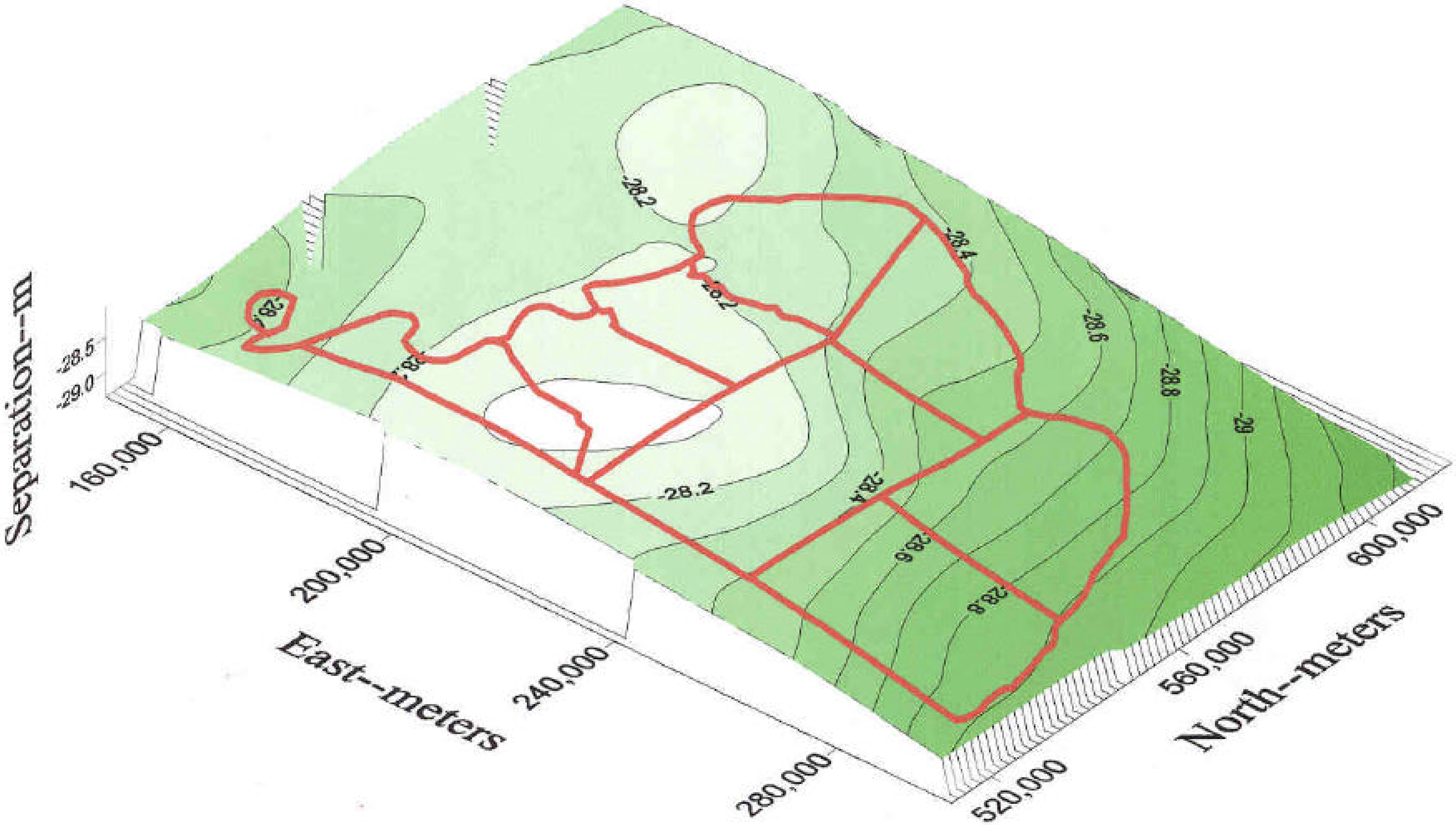

Figure 2. Model of ellipsoid-geoid separation (meters) in Jackson Purchase, Kentucky

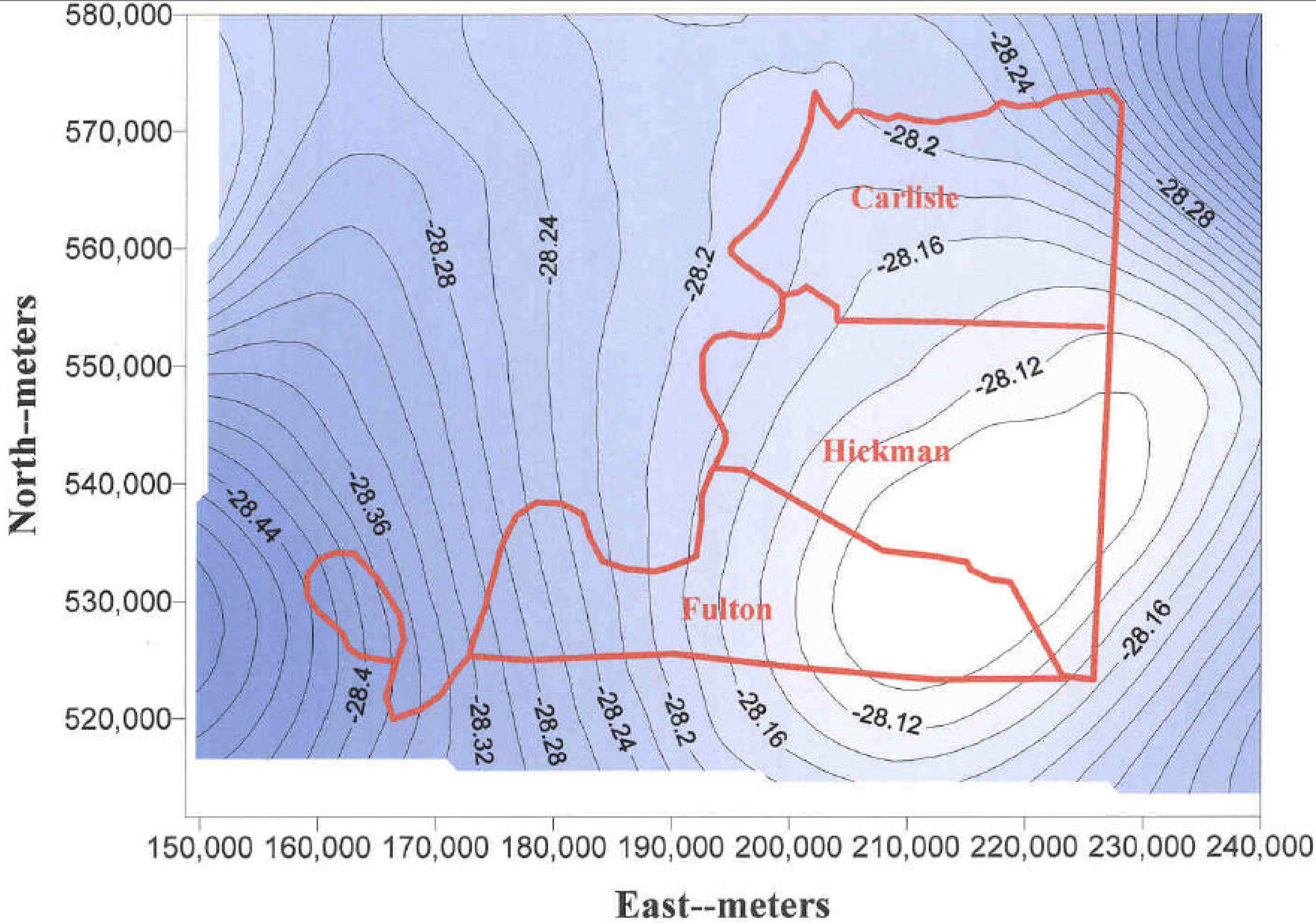

Figure 3. Ellipsoid-geoid separation (meters) in Carlisle, Hickman, & Fulton Counties, Kentucky

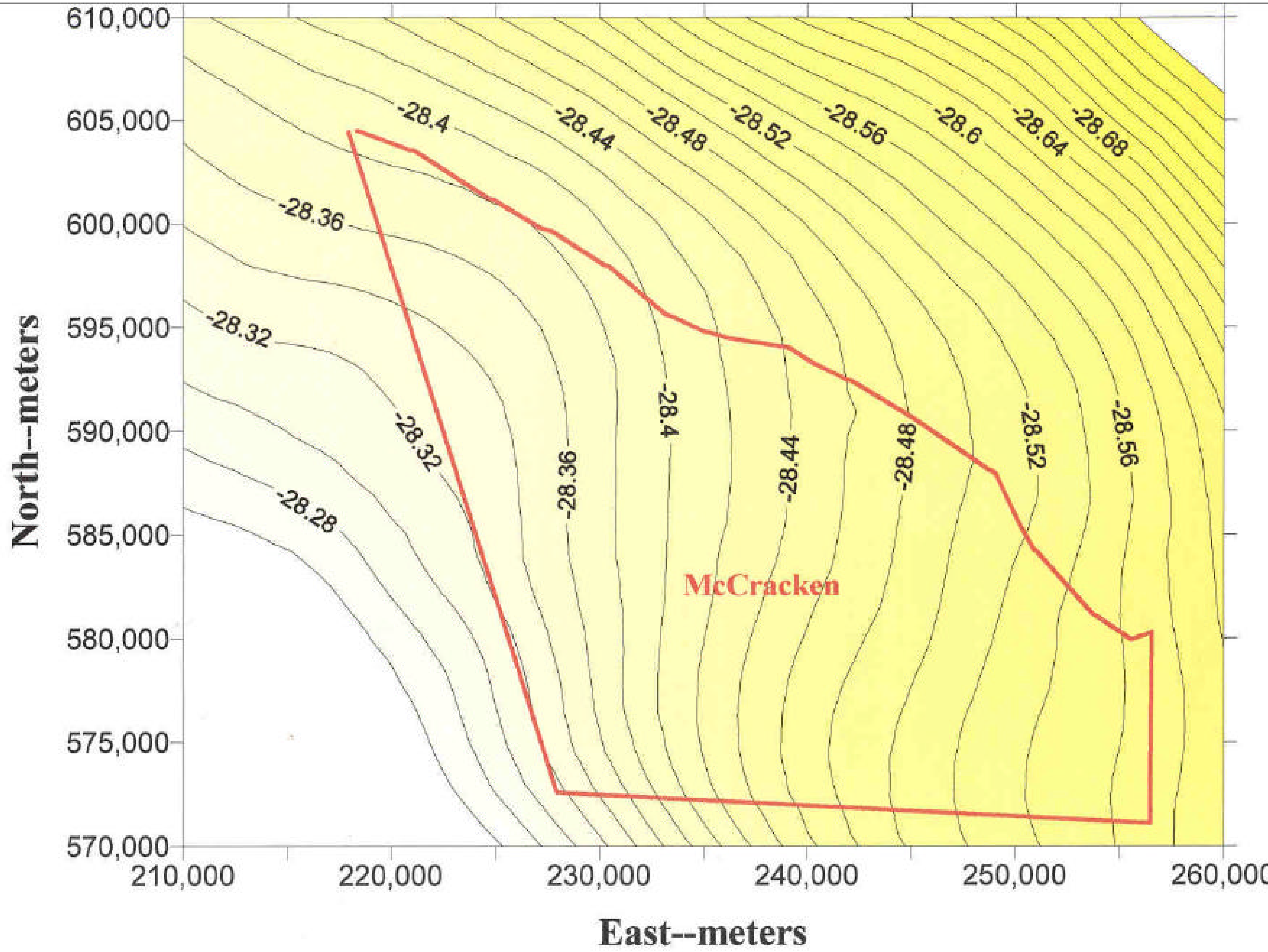

Figure 4. Ellipsoid-geoid separation (meters) in McCracken County, Kentucky

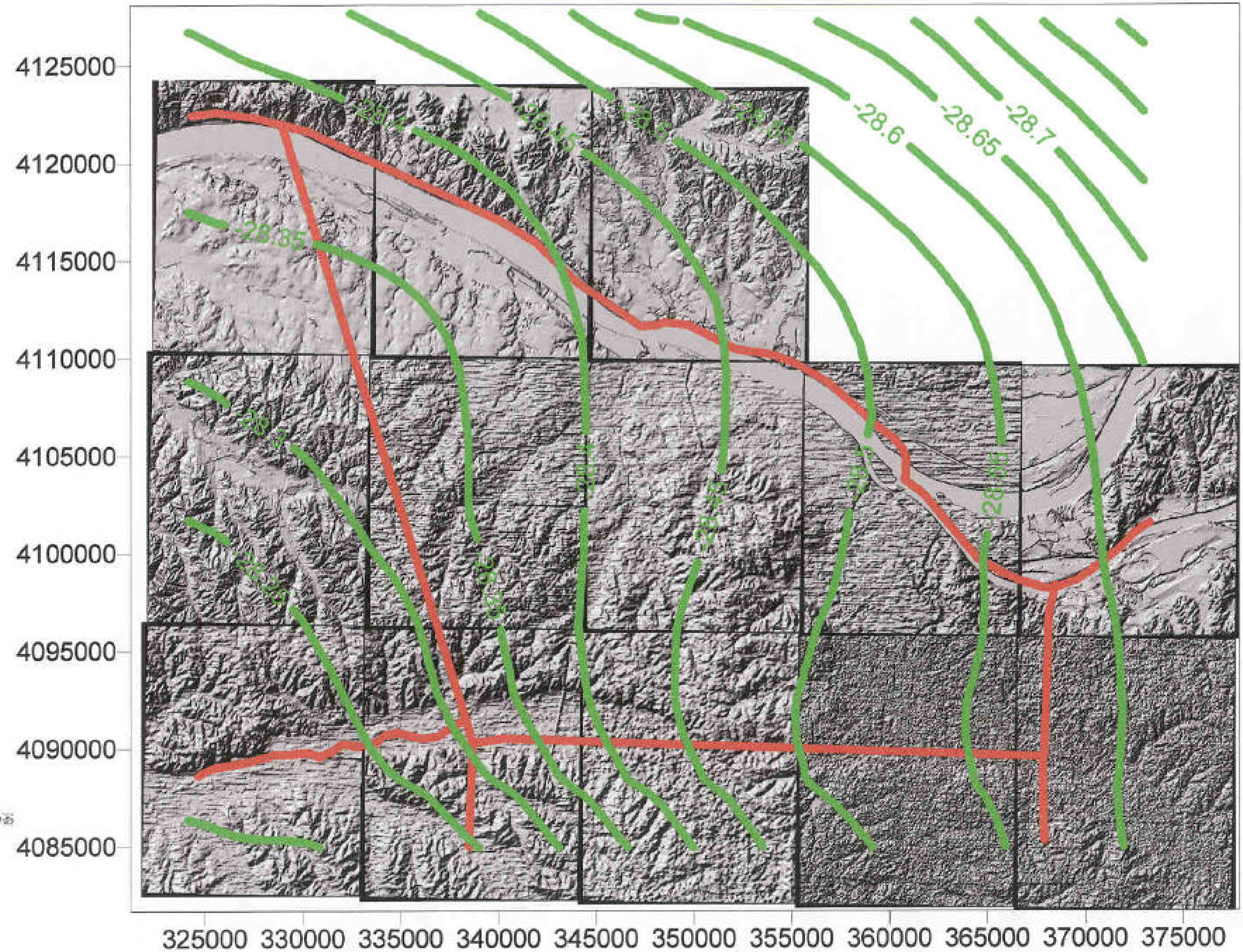

Fig. 5 Preliminary plot of ellipsoid-geoid separation, McCracken County, Kentucky