Basics of Two-Fluid Plasma Physics—a Short Summary

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The elementary building blocks for a multi-fluid are the canonical momentum $\mathbf{P}_\alpha = m_\alpha\mathbf{u}_\alpha + q_\alpha\mathbf{A}/c$, and the generalized vorticity $\boldsymbol{\Omega}_\alpha = \nabla\times\mathbf{P}_\alpha$ (or α-vorticity) where m_α, $\mathbf{u}_\alpha$, q_α are the species mass, flow velocity and charge, and $\alpha = i,e$ denotes the species, and $\mathbf{A}$ is the vector potential. The quadratic invariant of a species is the self helicity, or "α-helicity," the "density" of which is $\mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha$. These are generalizations of helicities that appear in a simple fluids and MHD. For zero electron mass the electron helicity reduces to the familiar magnetic helicity, an invariant in ideal MHD. The evolution of the α-helicities is governed by the helicity transport equation, derived from Maxwell's equations and the equations of motion.

Each helicity transport equation has the form, $n_\alpha D_\alpha(\mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha/n_\alpha)/Dt = \nabla\cdot[(\ldots)\boldsymbol{\Omega}_\alpha] +$ *friction*, where n_α is the density. The generalized vorticity appearing in the divergence term implies the existence of a "local" α-helicity associated with these lines, $K_\alpha = (c^2/8\pi q_\alpha^2)\int_C d\tau\,\mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha$, where C is the volume occupied by a bundle of α-vortex lines. The constant factor gives K_α the convenient units of energy-length. The total derivative D_α/Dt implies that the local α-helicity convects with its own species. If an α-vortex line does not intersect the system boundary, then in the strictly ideal (frictionless) case, the associated α-helicity is constant. There is a circulation theorem, $\Gamma_\alpha = \int_C \mathbf{P}_\alpha\cdot d\mathbf{x} = const$, where C is an α-vortex line, and $d\mathbf{x}$ is a differential length vector along that line. Each species has its own set of α-vortex lines, its own local α-helicities, and its own circulation theorem.

In the realistic case with friction, visco-resistive instabilities drive reconnections that break individual α-vortex lines and destroy their identity. This is a case of non-uniform convergence because even a minute amount of friction is enough to compromise the local α-helicities. The only quantities immune to these topology altering events are the global α-helicities, $K_\alpha = (c^2/8\pi q_\alpha^2)\int_V d\tau\,\mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha$, where V is the system volume. Even global invariants may not be *rugged* in the sense that they are *more* "invariant" than the organized energy form, *i.e*. the magnetofluid energy $W_{mf} = \int_V d\tau\,(\Sigma m_\alpha n_\alpha u_\alpha^2 + B^2/8\pi)$, composed of the flow energy and the magnetic energy (the sum is over species). The ruggedness of the global α-helicities has been supported by three arguments. (1) *Selective decay*: W_{mf} decays more rapidly than K_α in thin reconnection layers. Properly applied, this argument must account for limits on viscous friction coefficients for sharp gradients. (2) *Inverse cascade*: the fluctuation spectrum of $\tilde{W}_{mf}(k)$ and $\tilde{K}_\alpha(k)$ satisfy the necessary conditions for a cascade toward larger scale objects (k is the wave number of the disturbance). (3) *Stability to resistive modes*: K_α is less affected than W_{mf} by resistive modes. Each of these is the generalization of arguments previously applied to verify the ruggedness of the magnetic helicity in weakly-dissipative MHD.

A minimum energy state is found formally by minimizing W_{mf} subject ot invariant α-helicities, and (given axisymmetric system boundary) the global angular momentum, $L_\theta = \int d\tau\, r\Sigma m_\alpha n_\alpha u_{\alpha\theta}$. The variation with respect to $\delta\mathbf{u}_\alpha$ leads to the flow equations: $n_\alpha(\mathbf{u}_\alpha - \Omega r\hat{\boldsymbol{\theta}}) = (\lambda_\alpha/\ell_c^2)\boldsymbol{\Omega}_\alpha$ where λ_α, Ω are the Lagrange multipliers associated with invariant α-helicities and angular momentum, and $\ell_c = c/\omega_{pi} = (m_i c^2/4\pi e^2)^2$ is the length scale. An entropy maximization procedure subject to invariant K_α, L_θ, and total energy (W_{mf} + thermal) leads to the same equation. In addition, a global Bernoulli equation links the pressure to the flow by a relation that applies throughout the system volume. Note that an important feature of a two-fluid minimum energy state is the length scale ℓ_c. A two-fluid may or may not relax to the minimum energy state depending on whether the fast mechanisms have been stabilized.

Reference: L.C. Steinhauer and A. Ishida, Phys. Plasmas **5**, 2609 (1998)