Basics of Two-Fluid Equilibria—a Short Summary

Loren Steinhauer

The formalism is developed for axisymmetric multi-fluid equilibria as follows. First the various continuity equations governing the fields (Gauss's law of magnetism, steady Faraday's law) and flows (species continuity) are replaced in favor of scalar variables (stream functions, *etc*.).

$$\mathbf{B} = \hat{\boldsymbol{\theta}}\phi/r + (\hat{\boldsymbol{\theta}}\times\nabla\psi)/r\,; \qquad n_\alpha\mathbf{u}_\alpha = \hat{\boldsymbol{\theta}}\phi_\alpha/r + (\hat{\boldsymbol{\theta}}\times\nabla\psi_\alpha)/r$$

where $\mathbf{B}$ is the magnetic field, n_α and $\mathbf{u}_\alpha$ are the species density and velocity; ϕ, ϕ_α are the toroidal field and flow variables, and ψ, ψ_α are the field and flow stream functions. The remaining equations for the fields (electromagnetic, gravitational) can then be expressed in terms of these scalar variables, *e.g*. the toroidal and poloidal Ampere's laws are

$$\Delta^*\psi = \frac{4\pi}{c}\sum_\alpha q_\alpha\phi_\alpha\,; \qquad \phi = -\frac{4\pi}{c}\sum_\alpha q_\alpha\psi_\alpha$$

where $\Delta^* = \nabla\cdot[(1/r^2)\nabla]$ is the familiar Grad-Shafranov operator, and q_α is the charge of species α. The equation of motion for each species is simplified by two steps: (1) introduce the classical thermodynamic enthalpy $h_\alpha(p_\alpha,S_\alpha)$ where p_α,S_α are the pressure and entropy variables for species α; and (2) express the Lorentz force in terms of the generalized vorticity. These take particularly simple forms using the α-surface variable

$$\Psi_\alpha \quad = \quad \psi - (m_\alpha c/q_\alpha n_\alpha)\phi_\alpha$$

The surfaces $\Psi_\alpha = const$ are the drift surfaces for the α species. Then the equation of motion simplifies to

$$\nabla H_\alpha - \Theta_\alpha S'_\alpha\nabla\Psi_\alpha \quad = \quad \mathbf{u}_\alpha\times\boldsymbol{\Omega}_\alpha$$

where the *total enthalpy* is $H_\alpha = h_\alpha + m_\alpha u_\alpha^2/2 + q_\alpha V_E + m_\alpha V_G$, $\boldsymbol{\Omega}_\alpha = m_\alpha\nabla\times\mathbf{u}_\alpha + q_\alpha\mathbf{B}/c$ is the α-vorticity, and $\Theta_\alpha = \partial h_\alpha/\partial S_\alpha$. This generalizes the total enthalpy of a simple fluid in that it is species-specific and it also accounts for electrostatic (V_E) and gravitational (V_G) potential energies. Consideration of the three principal components of the equation of motion (θ, $\boldsymbol{\Omega}_\alpha$, $\nabla\Psi_\alpha$) leads to the identification of the three arbitrary surface functions for each species: entropy, $S_\alpha = S_\alpha(\Psi_\alpha)$; flow stream function, $\psi_\alpha = (c/4\pi q_\alpha)G_\alpha(\Psi_\alpha)$; and total enthalpy, $H_\alpha = H_\alpha(\Psi_\alpha)$. The result is a closed system of equations describing axisymmetric, multi-fluid equilibria.

The standard reduced case of a two-fluid assumes massless electrons, quasineutrality, and ideal species, $h_\alpha = [\gamma/(\gamma-1)]p_\alpha^{(\gamma-1)/\gamma}S_\alpha^{\;1/\gamma}$, where γ is the adiabatic index. This leads to a system of two second order equations for the magnetic and ion stream functions plus an auxiliary "Bernoulli" equation for the density.

$$\frac{nr^2}{G'_i}\nabla\cdot\left(\frac{G'_i}{nr^2}\nabla\Psi_i\right) \quad = \quad \frac{1}{\ell_c^2 G_i'^2}\left\{\frac{\psi-\Psi_i}{\ell_c^2} + (G_i+G_e)G'_i + \frac{r^2}{\ell_c^2}\frac{m_i c^2}{e^2}\left[H'_i - \Theta(n)S'_i\right]\right\}$$

$$\Delta^*\psi \quad = \quad \frac{\psi-\Psi_i}{\ell_c^2} - (G_i+G_e)G'_e - \frac{r^2}{\ell_c^2}\frac{m_i c^2}{e^2}\left[H'_e - \Theta(n)S'_e\right]$$

$$\frac{\gamma C_{st}}{\gamma-1}n^{\gamma-1}(\Psi_i+\psi)+\frac{1}{2r^2}\frac{e^2}{m_i c^2}\left[(\psi-\Psi_i)^2+\ell_c^4 G_i'^2\,|\nabla\Psi_i|^2\right] \;=\; H_i+H_e$$

The other variables are given by algebraic relations in terms of Ψ_i, ψ, and n:

$$\psi_i=\frac{c}{4\pi e}G_i(\Psi_i)\quad \psi_e=-\frac{c}{4\pi e}G_e(\psi)\quad \phi=-(G_i+G_e)\quad \phi_i=\frac{en}{m_i c}(\psi-\Psi_i)$$

This system reduces to the familiar Grad-Shafranov equation with the specification of of no ion flow.

A subset of two-fluid equilibria have minimum energy subject to constraints on the α-helicities and the mechanical angular momentum. In the minimum energy state the arbitrary functions must take specific forms

$$G_\alpha(\Psi_\alpha)=\lambda_\alpha\Psi_\alpha+const\,;\qquad S_\alpha(\Psi_\alpha)=const;\qquad H_\alpha(\Psi_\alpha)=-\Omega\frac{q_\alpha}{c}\Psi_\alpha+const$$

where λ_α, Ω are the Lagrange multipliers associated with the α-helicity and angular momentum invariants.

Reference: L.C. Steinhauer, "*Formalism for Multi-Fluid Equilibria with Flow*," submitted to Phys. Plasmas, Feb. 1999.