A Primer for Understanding Joint Kinetics

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Section 1: Introduction

The purpose of this course is to provide a review of the mathematical techniques used to derive kinetic data and to present implications of these techniques on data interpretation.

The course will begin by introducing the equations of motion $(S\ t = I\ a \quad S\ \mathbf{F} = m\mathbf{a})$ and will demonstrate how they have been used in clinical biomechanics to determine joint forces, moments and powers. The equations of motion will first be applied in the more traditional inverse dynamics approach. A second method using generalized coordinates will be introduced. This method directly couples the equations in order to determine the influence that the moment at one joint will have on the other anatomical segments. Although the generalized method has been used extensively in computer simulation and surgical decision making models, it has not been widely taught in post-graduate motion analysis programs.

Finally, the implications of data interpretation using both approaches will be discussed. This discussion will include data from clinical cases as well as their application in computer simulations.

From this tutorial, the participant will learn to use the generalized form of the equations of motion to derive the simple relationships between joint kinetics and kinematics and thus be better equipped to understand and interpret their current motion analysis output. In addition, because this approach serves as the foundation for computer simulation, the participants will be better prepared to understand the result of surgical decision making models.

Section 2: Joint Dynamics Principles and their Application

2.1 Traditional form of the equations of motion as used for determining joint kinetics

Most commercial motion analysis software uses the inverse dynamics approach to generate kinetic information including joint forces, moments and powers. A link segment model with rigid body segments is typically assumed. To compute kinetics, the position and orientation of each of the segments must be known, as well as the accelerations, anthropometric measures and external forces, i.e. ground reaction forces.

There are some limitations to interpreting data generated in the traditional method. Specifically, some problems arise in determining the true role of the joint moments, forces and powers in generating motion. A computer simulated case will be introduced and joint kinetics computed by inverse dynamics will be interpreted.

INVERSE DYNAMICS ANALYSIS

Goal: To apply inverse dynamics techniques to compute kinetic information, including joint reaction forces, moments, and power. A three link planar segment model will be evaluated for the purposes of this workshop.

Input:

1. Kinematic description of the body; joint positions (q), accelerations (**a**), angular accelerations ($\ddot{q}$), segment lengths (**l**) and center of gravity locations (**r**).
2. Anthropometric measures; mass moments of inertia (**I**)**,** masses (m).
3. External forces; ground reaction force, $\mathbf{F_G}$.

Output of Computations:

1. Joint reaction forces, **F**, and Moments, t.
2. Joint powers.
3. Muscle moments.

Assumptions:

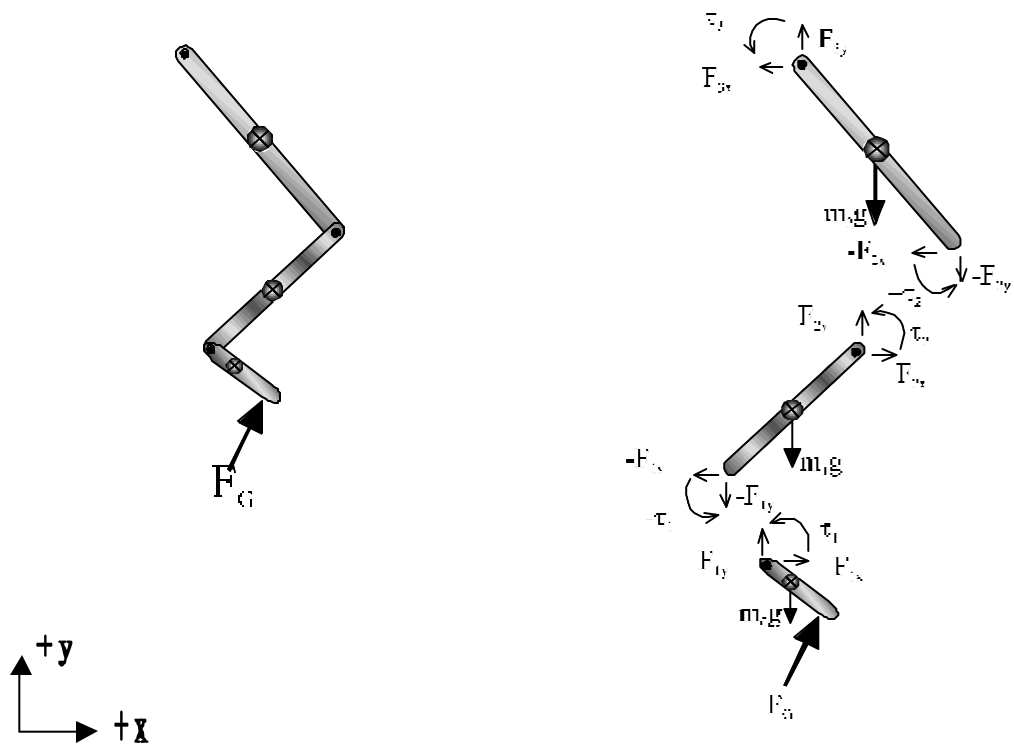
1. Each segment has a fixed mass whose location remains fixed at the Center of Mass, COM.
2. The mass moment of inertia for each segment remains constant.
3. The length of each segment remains constant.

The validity of the results is dependent on the model and on accurate measures of segment masses, centers of mass, joint centers, and mass moments of inertia.

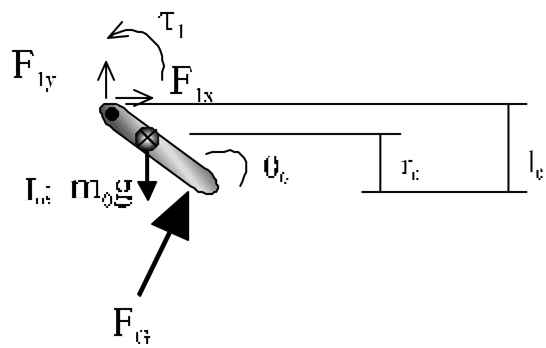
Newton's Law

$\Sigma \mathbf{F} = m\mathbf{a}$ $\qquad\qquad$ $\Sigma t = \mathbf{I}\,\ddot{q}$

Planar Link Segment Example



Link 0



$\Sigma\ \mathbf{F} = m_0\mathbf{a}_0$

$\Sigma\ \mathbf{F}_x = m_0\mathbf{a}_{0x}$
$\mathbf{F}_{1x} + \mathbf{F}_{Gx} = m_0\mathbf{a}_{0x}$
$\mathbf{F}_{1x} = m_0\mathbf{a}_{0x} - \mathbf{F}_{Gx}$

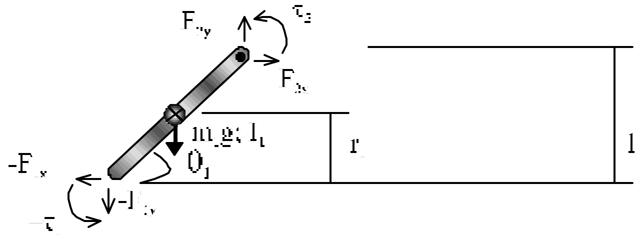
$\Sigma\ \mathbf{F}_y = m_0\mathbf{a}_{0y}$
$\mathbf{F}_{1y} - m_0\mathbf{g} + F_{Gy} = m_0\mathbf{a}_{0y}$
$\mathbf{F}_{1y} = m_0\mathbf{a}_{0y} + m_0\mathbf{g} - \mathbf{F}_{Gy}$

$\Sigma\ t = \mathbf{I}_0\ddot{q}_0$

$t_1 + (-\mathbf{r}_0 \times \mathbf{F}_G) + ((\mathbf{l}_0-\mathbf{r}_0) \times \mathbf{F}_1) = \mathbf{I}_0\ddot{q}_0$

$t_1 = \mathbf{I}_0\ddot{q}_0 - (-\mathbf{r}_0 \times \mathbf{F}_G) - ((\mathbf{l}_0-\mathbf{r}_0) \times \mathbf{F}_1) = \mathbf{I}_0\ddot{q}_0 - (-\mathbf{r}_{0x}\mathbf{F}_{Gy} + \mathbf{r}_0\mathbf{F}_{Gx}) - ((\mathbf{l}_0-\mathbf{r}_0)_x\mathbf{F}_{1y} - (\mathbf{l}_0-\mathbf{r}_0)_y\mathbf{F}_{1x})$

Link 1



$\Sigma \mathbf{F} = m_1\mathbf{a}_1$

$\Sigma \mathbf{F}_x = m_1\mathbf{a}_{1x}$
$\mathbf{F}_{1x} + \mathbf{F}_{2x} = m_1\mathbf{a}_{1x}$
$\mathbf{F}_{2x} = m_1\mathbf{a}_{1x} + \mathbf{F}_{1x}$
$\mathbf{F}_{2x} = m_1\mathbf{a}_{1x} + m_0\mathbf{a}_{0x} - \mathbf{F}_{Gx}$

$\Sigma \mathbf{F}_y = m_1\mathbf{a}_{1y}$
$\mathbf{F}_{2y} - \mathbf{F}_{1y} - m_1\mathbf{g} = m_1\mathbf{a}_{1y}$
$\mathbf{F}_{2y} = m_1\mathbf{a}_{1y} + \mathbf{F}_{1y} + m_1\mathbf{g}$
$\mathbf{F}_{2y} = m_1\mathbf{a}_{1y} + (m_0\mathbf{a}_{0y} + m_0\mathbf{g} - \mathbf{F}_{Gy}) + m_1\mathbf{g}$

$\Sigma t = \mathbf{I}_1\ddot{q}_1$

$t_2 - t_1 + (-\mathbf{r}_1 \text{ x } -\mathbf{F}_1) + ((\mathbf{l}_1-\mathbf{r}_1) \text{ x } \mathbf{F}_2) = \mathbf{I}_1\ddot{q}_1$

$t_2 = \mathbf{I}_1\ddot{q}_1 + t_1 - (-\mathbf{r}_1 \text{ x } -\mathbf{F}_1) - ((\mathbf{l}_1-\mathbf{r}_1) \text{ x } \mathbf{F}_2)$

$t_2 = \mathbf{I}_1\ddot{q}_1 + (\mathbf{I}_0\ddot{q}_0 - (\mathbf{r}_{0x}\mathbf{F}_{Gy} + \mathbf{r}_0\mathbf{F}_{Gx}) - ((\mathbf{l}_0-\mathbf{r}_0)_x\mathbf{F}_{1y} - (\mathbf{l}_0-\mathbf{r}_0)_y\mathbf{F}_{1x})) - (\mathbf{r}_{1x}\mathbf{F}_{1y} - \mathbf{r}_{1y}\mathbf{F}_{1x})$
$\quad - ((\mathbf{l}_1-\mathbf{r}_1)_x\mathbf{F}_{2y} - (\mathbf{l}_1-\mathbf{r}_1)_y\mathbf{F}_{2x})$

Joint powers can then be computed as the sum of proximal and distal segment power. This is equivalent to the product of net joint force and net joint velocity plus the product of net joint torque and net joint angular velocity.

$\mathbf{JP}_1 = \mathbf{P}_0 + \mathbf{P}_1$
$\mathbf{JP}_1 = (t_1\dot{q}_0) + (-t_1\dot{q}_1) + (\mathbf{F}_1\mathbf{V}_0) + (-\mathbf{F}_1\mathbf{V}_1) = t_1(\dot{q}_0 - \dot{q}_1)$

2.2 Limitations of the inverse dyamics approach

The limitations of the traditional inverse dynamics approach will be discussed and illustrated through a simple musculoskeletal model. These limitations can include:

- Inadequate explanation for the sources of the observed joint movements.
- Errors in determining the type of muscle contraction

2.3 Coupled dynamics - Generalized coordinate form of the equations of motion as used for determining joint kinetics

Generalized coordinates are a common engineering method used to reduce the equations of motion to one independent variable for each degree-of-freedom in a link model. As an example, the equations of motion will be derived for a simple two-link planar model using generalized coordinates. This example will be then used to illustrate the inherent relationships between joint kinetics and kinematics and demonstrate how each joint moment will produce accelerations at all of the joints in the body.

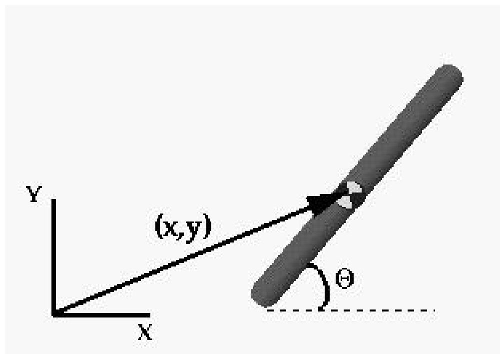
The resulting form of equations will be then be examined for their significance in understanding clinical motion analysis data. This technique will be applied to the computer simulation previously presented and reviewed in further detail to demonstrate the differences between the two sets of kinetic data.

References:

Zajac, F.E., Gordon, M.E. *Determining Muscle's Force and Action in Multi-Articular Movement*, Exer. Sport Sci. Rev 1989; 17: 187-230.

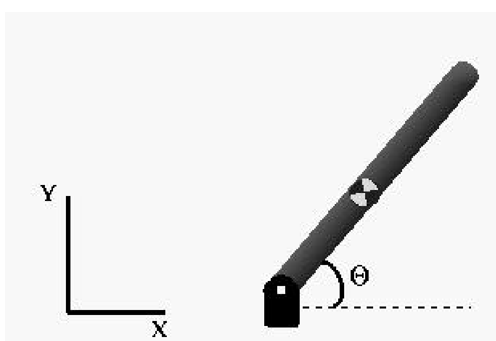
Generalized Coordinates: Minimum number of coordinates required to specify the configuration of the system. If each generalized coordinate can vary independently (holonomic system) then the number of generalized coordinates will equal the degrees of freedom of the system.

Planar example (no constraints):



The rigid body has 3 degrees of freedom (2 translational, 1 rotational); the configuration can be specified by three generalized coordinates **x**, **y**, q.

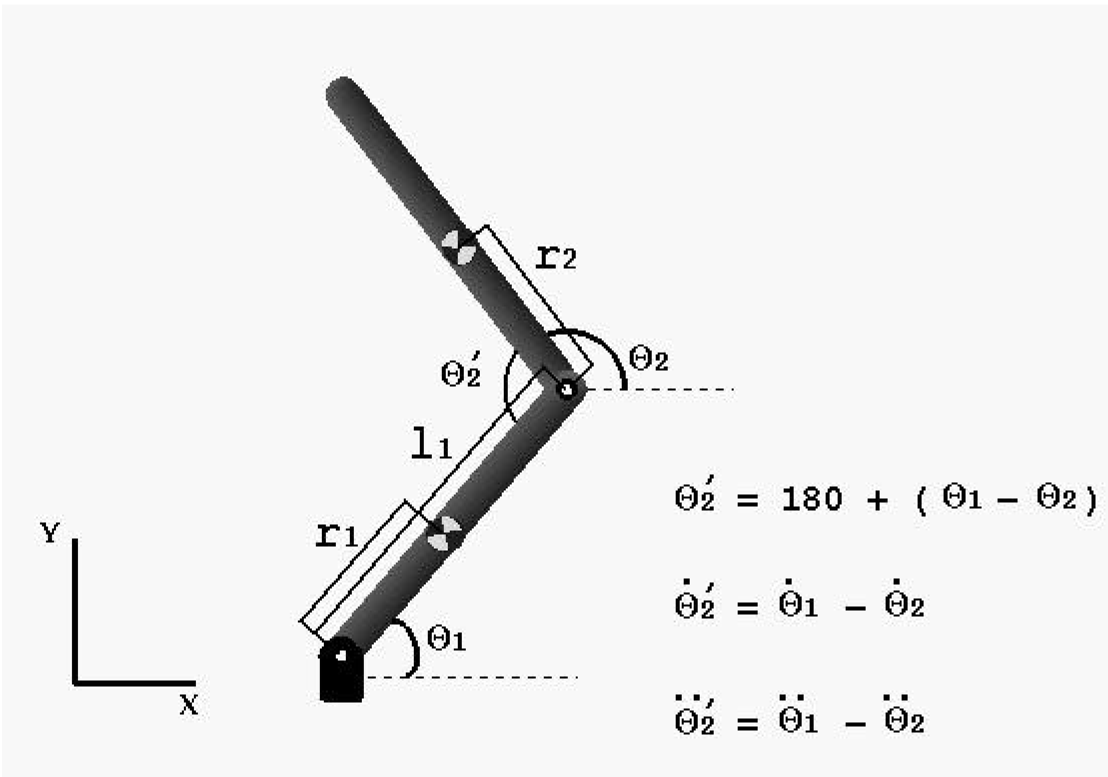
Planar example with constraint:



A translational constraint (pin joint) has been added. The rigid body has 1degrees of freedom (rotational); the configuration can be specified by the generalized coordinate q.

Example: Planar Two Link System

Goal: To find an expression for the joint moments such that the moments are a function of only the generalized coordinates (q_1, q_2) and their derivatives ($\dot{q}_1$, $\dot{q}_2$, $\ddot{q}_1$, $\ddot{q}_2$).



Later in the tutorial it will be shown that the generalized form of the joint moments t_1 and t_2 are:

$$t_1 = (I_1 + m_2r_2l_1\cos(\theta_1 - \theta_2) + m_1r_1^2 + m_2l_1^2)\,\ddot{q}_1$$

$$+ (I_2 + m_2r_2^2 + m_2r_2l_1\cos(\theta_1 - \theta_2)\ \ddot{q}_2$$

$$- \ m_2r_2l_1\sin(\theta_1 - \theta_2)(\dot{q}_1)^2$$

$$+ \ m_2r_2l_1\sin(\theta_1 - \theta_2)(\dot{q}_2)^2$$

$$+ (m_2r_2\cos\theta_2 + m_1r_1\cos\theta_1 + m_2l_1\cos\theta_1)\mathbf{g}$$

$$t_2 = m_2r_2\ l_1\cos(\theta_1 - \theta_2)\,\ddot{q}_1 + (I_2 + m_2r_2^2)\,\ddot{q}_2 - m_2r_2\ l_1\sin(\theta_1 - \theta_2)\,\dot{q}_1^2 + m_2r_2\cos\theta_2\mathbf{g}$$

For simplicity introduce the substitutions:

$M_{11} = I_1 + m_2 r_2 l_1 \cos(\theta_1 - \theta_2) + m_1 r_1^2 + m_2 l_1^2$

$M_{12} = (I_2 + m r_2^2 + m_2 r_2 l_1 \cos(\theta_1 - \theta_2)$

$M_{21} = m_2 r_2\, l_1 \cos(\theta_1 - \theta_2)$

$M_{22} = (I_2 + m_2 r_2^2)$

$C_{11} = -m_2 r_2 l_1 \sin(\theta_1 - \theta_2)\, (\dot{q}_1)^2$

$C_{12} = m_2 r_2 l_1 \sin(\theta_1 - \theta_2)\, (\dot{q}_2)^2$

$C_{21} = -m_2 r_2 l_1 \sin(\theta_1 - \theta_2)\, (\dot{q}_1)^2$

$C_{22} = 0$

$G_{11} = m_2 r_2 \cos\theta_2 + m_1 r_1 \cos\theta_1 + m_2 l_1 \cos\theta_1 \mathbf{g}$

$G_{21} = m_2 r_2 \cos\theta_2\, \mathbf{g}$

Thus:

$t_1 = M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 - C_{11} + C_{12} + G_{11}$

$t_2 = M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 - C_{21} + C_{22} + G_{21}$

Reformulate as matrix algebra:

$$\begin{bmatrix} \tau_{11} \\ \tau_{21} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} + \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix}$$

The joint moments can be expressed in matrix notation:

$$\tau = M(\theta)\ddot{q} + C(\theta, \dot{\theta}) + G(\theta)$$

Solving for $\ddot{q}$:

$$M(\theta)\ddot{q} = \tau - C(\theta, \dot{\theta}) - G(\theta)$$

$$\ddot{q} = M^{-1}(\theta)\tau - M^{-1} C(\theta, \dot{\theta}) - M^{-1} G(\theta)$$

Implications for Gait Analysis

The following principals of joint kinetics can be uncovered:

1. A moment at a joint will act to accelerate all of the joints of the body.
2. The magnitude of the accelerations created by a joint moment will be a function of both the magnitude of the moment and the positions of the segments.
3. The accelerations created by a joint moment are independent of the velocity of the joint. Thus, the accelerations created by a given joint moment are independent of type of contraction (eccentric vs. concentric.)

2.4 Clinical case and examples

A clinical case will be used to compare the traditional and generalized methods. The clinical case will demonstrate how the generalized method can be used to directly measure the compensatory mechanics used in a patient with lower extremity weakness.

2.5 Discussion and questions

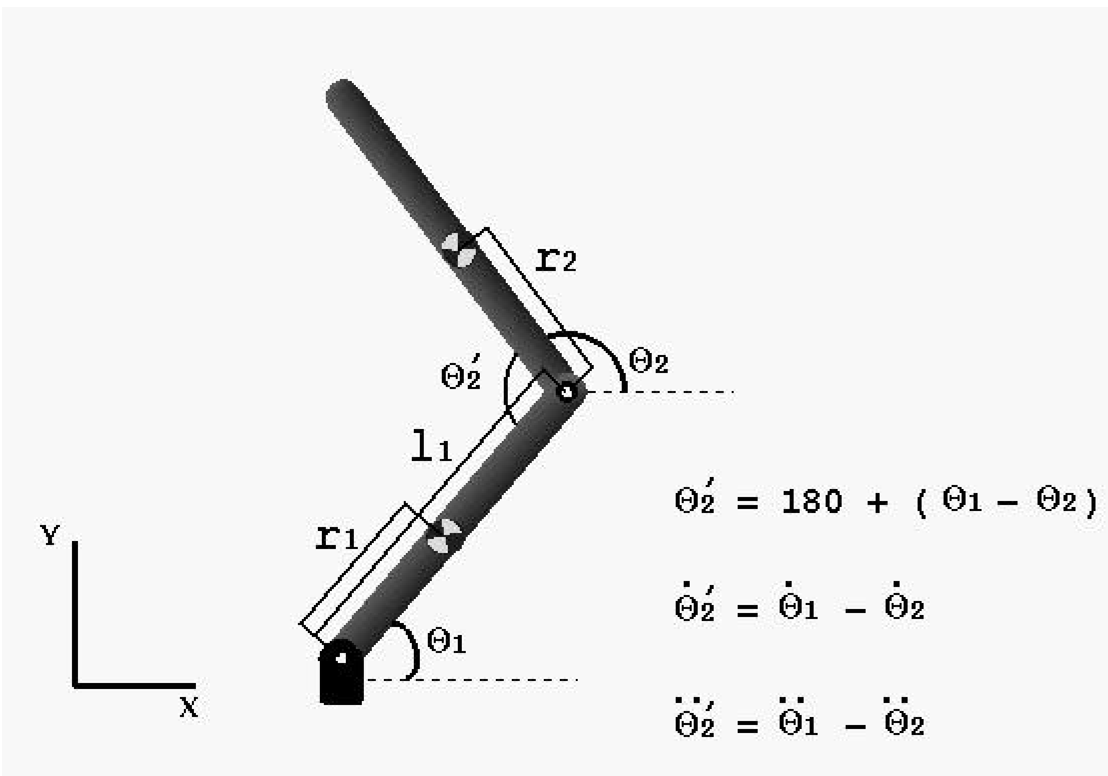
We hope to initiate discussion among participants who are involved in clinical interpretation of kinetic data as well as those involved in forward dynamic modeling.

Section 3: Computation of the Net Joint Dynamics

3.1 Derivation of the Joint Moment Equations (Generalized Form)

Example: Planar Two Link System

Goal: To find an expression for the joint moments such that the moments are a function of only the generalized coordinates (q_1, q_2) and their derivatives ($\dot{q}_1, \dot{q}_2, \ddot{q}_1, \ddot{q}_2$).



Link 2

Basic Equations :

Equations of Motion: $S\ t = I\ddot{q}$ $\quad$ $S\ \mathbf{F} = m\mathbf{a}$

$$t_2 + (-r_2 \text{ x } \mathbf{F}_2) = I_2\ddot{q}_2 \qquad \mathbf{F}_2 - m_2\mathbf{g} = m_2\mathbf{a}_{2cg}$$

Trig relationships:

$$\mathbf{r}_{2x} = r_2\cos\theta_2 \qquad \mathbf{r}_{2y} = r_2\sin\theta_2$$
$$\mathbf{l}_{1x} = l_1\cos\theta_1 \qquad \mathbf{l}_{1y} = l_1\sin\theta_1$$

Relative Motion Equations: $\mathbf{a}_{2cg} = \mathbf{a}_{2\_joint} + \mathbf{a}_{2\_tangent} + \mathbf{a}_{2\_normal}$

where:

$$\mathbf{a}_{2\_tangent} = \ddot{q}_2 \text{ x } \mathbf{r}_2$$
$$\mathbf{a}_{2\_normal} = \dot{q}_2 \text{ x } (\dot{q}_2 \text{ x } \mathbf{r}_2)$$
$$\mathbf{a}_{2\_joint} = \ddot{q}_1 \text{ x } \mathbf{l}_1 + \dot{q}_1 \text{ x } (\dot{q}_1 \text{ x } \mathbf{l}_1)$$

thus:

$$\mathbf{a}_{2cg} = \ddot{q}_1 \text{ x } \mathbf{l}_1 + \dot{q}_1 \text{ x } (\dot{q}_1 \text{ x } \mathbf{l}_1) + \ddot{q}_2 \text{ x } \mathbf{r}_2 + \dot{q}_2 \text{ x } (\dot{q}_2 \text{ x } \mathbf{r}_2)$$

cross product terms:

x terms	**y terms**
$(\ddot{q}_1 \text{ x } \mathbf{l}_1)_x = -\mathbf{l}_{1y}\ddot{q}_1$	$(\ddot{q}_1 \text{ x } \mathbf{l}_1)_y = \mathbf{l}_{1x}\ddot{q}_1$
$(\dot{q}_1 \text{ x } (\dot{q}_1 \text{ x } \mathbf{l}_1))_x = -\mathbf{l}_{1x}(\dot{q}_1)^2$	$(\dot{q}_1 \text{ x } (\dot{q}_1 \text{ x } \mathbf{l}_1))_y = -\mathbf{l}_{1y}(\dot{q}_1)^2$
$(\ddot{q}_2 \text{ x } \mathbf{r}_2)_x = -\mathbf{r}_{2y}\ddot{q}_2$	$(\ddot{q}_2 \text{ x } \mathbf{r}_2)_y = \mathbf{r}_{2x}\ddot{q}_2$
$(\dot{q}_2 \text{ x } (\dot{q}_2 \text{ x } \mathbf{r}_2))_x = -\mathbf{r}_{2x}(\dot{q}_2)^2$	$(\dot{q}_2 \text{ x } (\dot{q}_2 \text{ x } \mathbf{r}_2))_y = -\mathbf{r}_{2y}(\dot{q}_2)^2$

collecting the terms:

$$(\mathbf{a}_{2cg})_x = -\mathbf{l}_{1y}\ddot{q}_1 + -\mathbf{l}_{1x}(\dot{q}_1)^2 + -\mathbf{r}_{2y}\ddot{q}_2 + -\mathbf{r}_{2y}(\dot{q}_2)^2$$
$$= -l_1\sin\theta_1\ddot{q}_1 + -l_1\cos\theta_1(\dot{q}_1)^2 + -r_2\sin\theta_2\ddot{q}_2 + -r_2\cos\theta_2(\dot{q}_2)^2$$
$$(\mathbf{a}_{2cg})_y = \mathbf{l}_{1x}\ddot{q}_1 + -\mathbf{l}_{1y}(\dot{q}_1)^2 + \mathbf{r}_{2x}\ddot{q}_2 + \mathbf{r}_{2y}(\dot{q}_2)^2$$
$$= l_1\cos\theta_1\ddot{q}_1 + -l_1\sin\theta_1(\dot{q}_1)^2 + r_2\cos\theta_2\ddot{q}_2 + -r_2\sin\theta_2(\dot{q}_2)^2$$

- -

Find the torque at joint 2:

$$t_2 + (-r_2 \text{ x } \mathbf{F}_2) = I_2\ddot{q}_2$$

$$t_2 = I_2\ddot{q}_2 - (-\mathbf{r}_2 \text{ x } \mathbf{F}_2)$$

By vector algebra:

$$t_2 = \mathbf{I}_2\ddot{q}_2 + r_{2x}F_{2y} - r_{2y}F_{2x}$$

$$t_2 = I_2\ddot{q}_2 + r_2\cos\theta_2 F_{2y} - r_2\sin\theta_2 F_{2x}$$

Find the force at joint 2 ($\mathbf{F}_2$):

$$\mathbf{F}_2 - m_2\mathbf{g} = m_2\mathbf{a}_{2cg}$$

$$\mathbf{F}_2 = m_2\mathbf{a}_{2cg} + m_2\mathbf{g}$$

Find the components of $\mathbf{F}_2$:

$$F_{2x} = m_2(\mathbf{a}_{2cg})_x$$

$$F_{2x} = m_2(-l_1\sin\theta_1\ddot{q}_1 + -l_1\cos\theta_1(\dot{q}_1)^2 + -r_2\sin\theta_2\ddot{q}_2 + -r_2\cos\theta_2(\dot{q}_2)^2)$$

$$F_{2y} = m_2(\mathbf{a}_{2cg})_y + m_2\mathbf{g}$$

$$F_{2y} = m_2(l_1\cos\theta_1\ddot{q}_1 + -l_1\sin\theta_1(\dot{q}_1)^2 + r_2\cos\theta_2\ddot{q}_2 + -r_2\sin\theta_2(\dot{q}_2)^2 + \mathbf{g})$$

Substituting F_{2y} and F_{2x} into t_2:

$$t_2 = I_2\ddot{q}_2 + m_2r_2\cos\theta_2[l_1\cos\theta_1\ddot{q}_1 + -l_1\sin\theta_1(\dot{q}_1)^2 + r_2\cos\theta_2\ddot{q}_2 + r_2\sin\theta_2(\dot{q}_2)^2 + \mathbf{g}]$$
$$- m_2r_2\sin\theta_2[-l_1\sin\theta_1\ddot{q}_1 + -l_1\cos\theta_1(\dot{q}_1)^2 + -r_2\sin\theta_2\ddot{q}_2 + -r_2\cos\theta_2(\dot{q}_2)^2]$$

Collecting like terms:

$$t_2 = (I_2 + (m_2r_2\cos\theta_2)^2 + (m_2r_2\sin\theta_2)^2)\ddot{q}_2$$
$$+ m_2r_2 l_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\ddot{q}_1$$
$$+ m_2r_2 l_1(-\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\dot{q}_1^2$$
$$+ m_2r_2^2(-\sin\theta_2 \cos\theta_2 + \sin\theta_2 \cos\theta)\dot{q}_2^2$$
$$+ m_2r_2\cos\theta_2\mathbf{g}$$

Finally via algebra and trig:

$$t_2 = (I_2 + m_2r_2^2)\ddot{q}_2 + m_2r_2 l_1\cos(\theta_1 - \theta_2)\ddot{q}_1 - m_2r_2 l_1\sin(\theta_1 - \theta_2)\dot{q}_1^2 + m_2r_2\cos\theta_2\mathbf{g}$$

Link 1

Basic Equations:

Equations of Motion: $S\ t = I\ddot{q}$ $\qquad$ $S\ \mathbf{F} = m\mathbf{a}$

$$t_1 + (-t_2) + (-r_1 \times \mathbf{F}_1) + ((\mathbf{l}_1 - \mathbf{r}_1) \times (-\mathbf{F}_2)) = I_1\ddot{q}_1 \qquad \mathbf{F}_1 + (-\mathbf{F}_2) - m_1\mathbf{g} = m_1\mathbf{a}_{2cg}$$

Trig relationships: $\mathbf{r}_{1x} = r_1\cos\theta_1$ $\qquad$ $\mathbf{r}_{1y} = r_1\sin\theta_1$

$\mathbf{l}_{1x} = l_1\cos\theta_1$ $\qquad$ $\mathbf{l}_{1y} = l_1\sin\theta_1$

Relative Motion Equations: $\mathbf{a}_{1cg} = \mathbf{a}_{1\_joint} + \mathbf{a}_{1\_tangent} + \mathbf{a}_{1\_normal}$

where: $\mathbf{a}_{1\_tangent} = \ddot{q}_1 \times \mathbf{r}_1$

$\mathbf{a}_{1\_normal} = \dot{q}_1 \times (\dot{q}_1 \times \mathbf{r}_1)$

$\mathbf{a}_{1\_joint} = 0$

thus: $\mathbf{a}_{1cg} = \ddot{q}_{1,} \times \mathbf{r}_1 + \dot{q}_1 \times (\dot{q}_1 \times \mathbf{r}_1)$

cross product terms:

x terms

$(\ddot{q}_1 \times \mathbf{r}_1)_x = -\mathbf{r}_{1y}\ddot{q}_1$

$(\dot{q}_1 \times (\dot{q}_1 \times \mathbf{r}_1))_x = -\mathbf{r}_{1x}(\dot{q}_1)^2$

y terms

$(\ddot{q}_1 \times \mathbf{r}_1)_y = \mathbf{r}_{1x}\ddot{q}_1$

$(\dot{q}_1 \times (\dot{q}_1 \times \mathbf{r}_1))_y = -\mathbf{r}_{1y}(\dot{q}_1)^2$

collecting the terms:

$$(\mathbf{a}_{2cg})_x = -\mathbf{r}_{1y}\ddot{q}_1 + -\mathbf{r}_{1x}(\dot{q}_1)^2$$
$$= -r_1\sin\theta_1\ddot{q}_1 + -r_1\cos\theta_1(\dot{q}_1)^2$$
$$(\mathbf{a}_{2cg})_y = \mathbf{r}_{1x}\ddot{q}_1 + \mathbf{r}_{1y}(\dot{q}_1)^2$$
$$= r_1\cos\theta_1\ddot{q}_1 + -r_1\sin\theta_1(\dot{q}_1)^2$$

The torque at joint 1:

$$t_1 + (-t_2) + (-r_1 \times \mathbf{F}_1) + ((\mathbf{l}_1 - \mathbf{r}_1) \times (-\mathbf{F}_2)) = I_1\ddot{q}_1$$
$$t_1 = I_1\ddot{q}_1 + t_2 - (-r_1 \times \mathbf{F}_1) - ((\mathbf{l}_1 - \mathbf{r}_1) \times (-\mathbf{F}_2))$$

By vector algebra:

$$t_1 = I_1\ddot{q}_1 + t_2 + (r_1 \times \mathbf{F}_1) + ((\mathbf{l}_1 - \mathbf{r}_1) \times (\mathbf{F}_2))$$

$$t_1 = I_1\ddot{q}_1 + t_2 + (r_1 \times \mathbf{F}_1) + (\mathbf{l}_1 \times \mathbf{F}_2) - (\mathbf{r}_1 \times \mathbf{F}_2)$$

$$t_1 = I_1\ddot{q}_1 + t_2 + \mathbf{r}_{1x}\mathbf{F}_{1y} - \mathbf{r}_{1y}\mathbf{F}_{1x} + \mathbf{l}_{1x}\mathbf{F}_{2y} - \mathbf{l}_{1y}\mathbf{F}_{2x} - \mathbf{r}_{1x}\mathbf{F}_{2y} + \mathbf{r}_{1y}\mathbf{F}_{2x}$$

Collecting like terms:

$$t_1 = I_1\ddot{q}_1 + t_2 + \mathbf{r}_{1x}(\mathbf{F}_{1y} - \mathbf{F}_{2y}) - \mathbf{r}_{1y}(\mathbf{F}_{1x} - \mathbf{F}_{2x}) + \mathbf{l}_{1x}\mathbf{F}_{2y} - \mathbf{l}_{1y}\mathbf{F}_{2x}$$

Find expression for $\mathbf{F}_1 - \mathbf{F}_2$:

$$\mathbf{F}_1 + (-\mathbf{F}_2) - m_1\mathbf{g} = m_1\mathbf{a}_{2cg}$$

$$\mathbf{F}_1 - \mathbf{F}_2 = m_1\mathbf{a}_{2cg} + m_1\mathbf{g}$$

Find the components for $\mathbf{F}_1 - \mathbf{F}_2$:

$$\mathbf{F}_{1x} - \mathbf{F}_{2x} = m_1(\mathbf{a}_{2cg})_x$$

$$\mathbf{F}_{1y} - \mathbf{F}_{2y} = m_1(\mathbf{a}_{2cg})_y + m_1\mathbf{g}$$

Substituting $(\mathbf{F}_{1x} - \mathbf{F}_{2x})$ and $(\mathbf{F}_{1y} - \mathbf{F}_{2y})$ into t_1:

$$t_1 = I_1\ddot{q}_1 + t_2 + \mathbf{r}_{1x}(m_1(\mathbf{a}_{2cg})_y + m_1\mathbf{g}) - \mathbf{r}_{1y}(m_1(\mathbf{a}_{2cg})_x) + \mathbf{l}_{1x}\mathbf{F}_{2y} - \mathbf{l}_{1y}\mathbf{F}_{2x}$$

Substituting for $\mathbf{r}_{1x}$, $\mathbf{r}_{1y}$, $\mathbf{l}_{1x}$, and $\mathbf{l}_{1y}$:

$$t_1 = I_1\ddot{q}_1 + t_2 + m_1r_1\cos\theta_1((\mathbf{a}_{2cg})_y + \mathbf{g}) - m_1r_1\sin\theta_1(\mathbf{a}_{2cg})_x + l_1\cos\theta_1\mathbf{F}_{2y} - l_1\sin\theta_1\mathbf{F}_{2x}$$

By algebra:

$$t_1 = I_1\ddot{q}_1 + t_2 + m_1r_1\cos\theta_1(\mathbf{a}_{2cg})_y + m_1r_1\cos\theta_1\mathbf{g} - m_1r_1\sin\theta_1(\mathbf{a}_{2cg})_x$$
$$+ l_1\cos\theta_1\mathbf{F}_{2y} - l_1\sin\theta_1\mathbf{F}_{2x}$$

Substituting for $(\mathbf{a}_{2cg})_y$ and $(\mathbf{a}_{2cg})_x$ from Relative Motion Equations:

$$t_1 = I_1\ddot{q}_1 + t_2 + m_1r_1\cos\theta_1(r_1\cos\theta_1\ddot{q}_1 + -r_1\sin\theta_1(\dot{q}_1)^2) + m_1r_1\cos\theta_1\mathbf{g}$$
$$- m_1r_1\sin\theta_1(-r_1\sin\theta_1\ddot{q}_1 + -r_1\cos\theta_1(\dot{q}_1)^2) + l_1\cos\theta_1\mathbf{F}_{2y} - l_1\sin\theta_1\mathbf{F}_{2x}$$

Use the results of link 2 to substitute for t_2, $\mathbf{F}_{2x}$, $\mathbf{F}_{2y}$:

$$t_1 = I_1\ddot{q}_1$$
$$+ (I_2 + m_2r_2^2)\ddot{q}_2 + m_2r_2 l_1\cos(\theta_1 - \theta_2)\ddot{q}_1 - m_2r_2 l_1\sin(\theta_1 - \theta_2)\dot{q}_1^2 + m_2r_2\cos\theta_2\mathbf{g}$$
$$+ m_1r_1\cos\theta_1(r_1\cos\theta_1\ddot{q}_1 + -r_1\sin\theta_1(\dot{q}_1)^2) + m_1r_1\cos\theta_1\mathbf{g}$$
$$- m_1r_1\sin\theta_1(-r_1\sin\theta_1\ddot{q}_1 + -r_1\cos\theta_1(\dot{q}_1)^2)$$
$$+ l_1\cos\theta_1(m_2(l_1\cos\theta_1\ddot{q}_1 + -l_1\sin\theta_1(\dot{q}_1)^2 + r_2\cos\theta_2\ddot{q}_2 + -r_2\sin\theta_2(\dot{q}_2)^2 + \mathbf{g}))$$
$$- l_1\sin\theta_1(m_2(-l_1\sin\theta_1\ddot{q}_1 + -l_1\cos\theta_1(\dot{q}_1)^2 + -r_2\sin\theta_2\ddot{q}_2 + -r_2\cos\theta_2(\dot{q}_2)^2))$$

Collecting like terms:

$$t_1 = (I_1 + m_2r_2l_1\cos(\theta_1 - \theta_2) + m_1r_1^2\cos^2\theta_1 + m_1r_1^2\sin^2\theta_1 + m_2l_1^2\cos^2\theta_1 + m_2l_1^2\sin^2\theta_1)\ddot{q}_1$$
$$+ (I_2 + m_2r_2^2 + m_2r_2l_1\cos\theta_1\cos\theta_2 + m_2r_2l_1\sin\theta_1\sin\theta_2)\ddot{q}_2$$
$$+ (-m_2r_2l_1\sin(\theta_1 - \theta_2) - m_1r_1^2\sin\theta_1\cos\theta_1 + m_1r_1^2\sin\theta_1\cos\theta_1 - m_2l_1^2\sin\theta_1\cos\theta_1 + m_2l_1^2\sin\theta_1\cos\theta_1)(\dot{q}_1)^2$$
$$+ (-m_2r_2 l_1\cos\theta_1\sin\theta_2 + m_2r_2 l_1\sin\theta_1\cos\theta_2)(\dot{q}_2)^2$$
$$+ (m_2r_2\cos\theta_2 + m_1r_1\cos\theta_1 + m_2l_1\cos\theta_1)\mathbf{g}$$

Simplifying:

$$t_1 = (I_1 + m_2r_2l_1\cos(\theta_1 - \theta_2) + m_1r_1^2 + m_2l_1^2)\ddot{q}_1$$
$$+ (I_2 + m_2r_2^2 + m_2r_2l_1\cos(\theta_1 - \theta_2)\ \ddot{q}_2$$
$$- m_2r_2l_1\sin(\theta_1 - \theta_2)(\dot{q}_1)^2$$
$$+ m_2r_2l_1\sin(\theta_1 - \theta_2)(\dot{q}_2)^2$$
$$+ (m_2r_2\cos\theta_2 + m_1r_1\cos\theta_1 + m_2l_1\cos\theta_1)\mathbf{g}$$

Restating our expressions for t_1 and t_2:

$$t_1 = (I_1 + m_2r_2l_1\cos(\theta_1 - \theta_2) + m_1r_1^2 + m_2l_1^2)\,\ddot{q}_1$$
$$+ (I_2 + mr_2^2 + m_2r_2l_1\cos(\theta_1 - \theta_2)\;\ddot{q}_2$$
$$- \; m_2r_2l_1\sin(\theta_1 - \theta_2)(\dot{q}_1)^2$$
$$+ \; m_2r_2l_1\sin(\theta_1 - \theta_2)(\dot{q}_2)^2$$
$$+ (m_2r_2\cos\theta_2 + m_1r_1\cos\theta_1 + m_2l_1\cos\theta_1)\mathbf{g}$$

$$t_2 = m_2r_2\,l_1\cos(\theta_1 - \theta_2)\,\ddot{q}_1 + (I_2 + m_2r_2^2)\,\ddot{q}_2 - m_2r_2\,l_1\sin(\theta_1 - \theta_2)\,\dot{q}_1^2 + m_2r_2\cos\theta_2\mathbf{g}$$

For simplicity introduce the substitutions:

$M_{11} = I_1 + m_2r_2l_1\cos(\theta_1 - \theta_2) + m_1r_1^2 + m_2l_1^2$

$M_{12} = (I_2 + mr_2^2 + m_2r_2l_1\cos(\theta_1 - \theta_2)$

$M_{21} = m_2r_2\,l_1\cos(\theta_1 - \theta_2)$

$M_{22} = (I_2 + m_2r_2^2)$

$C_{11} = -m_2r_2l_1\sin(\theta_1 - \theta_2)\,(\dot{q}_1)^2$

$C_{12} = m_2r_2l_1\sin(\theta_1 - \theta_2)\,(\dot{q}_2)^2$

$C_{21} = -m_2r_2l_1\sin(\theta_1 - \theta_2)\,(\dot{q}_1)^2$

$C_{22} = 0$

$G_{11} = m_2r_2\cos\theta_2 + m_1r_1\cos\theta_1 + m_2l_1\cos\theta_1\,\mathbf{g}$

$G_{21} = m_2r_2\cos\theta_2\,\mathbf{g}$

Thus:

$$t_1 = M_{11}\,\ddot{q}_1 + M_{12}\ddot{q}_2 - C_{11} + C_{12} + G_{11}$$
$$t_2 = M_{21}\,\ddot{q}_1 + M_{22}\ddot{q}_2 - C_{21} + C_{22} + G_{21}$$

Reformulate as matrix algebra:

$$\begin{bmatrix}\tau_{11}\\ \tau_{21}\end{bmatrix} = \begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix} + \begin{bmatrix}G_{11}\\ G_{21}\end{bmatrix}$$

Matrix Notation:

$$\tau = M(\theta)\ddot{q} + C(\theta, \dot{\theta}) + G(\theta)$$

Solve for $\ddot{q}$:

$$M(\theta)\ddot{q} = \tau - C(\theta,\dot{\theta}) - G(\theta)$$

$$\ddot{q} = M^{-1}(\theta)\tau - M^{-1} C(\theta,\dot{\theta}) - M^{-1}G(\theta)$$

3.2 Basis for Computer Simulation

The generalized form of the equations of motion can be used as the basis for computer simulation. Computer simulation is conducted by specifying initial values for the joint positions and velocities (the state variables) and solving for the accelerations using the equation above. Numerical integration can then be applied to the accelerations in order to predict the state variables (joint positions and velocities) for a small time forward in time. After obtaining the new state variables, the equations of motion are then used to solve for the new acceleration. This process is repeated to produce simulated movements.

References:

Nikravesh, P.E. Computer-Aided Analysis of Mechanical Systems. Prentice Hall, 1988.

Appendix

Equations of Motion: $S\ t = I\ddot{q}$ $\qquad$ $S\ \mathbf{F} = m\mathbf{a}$

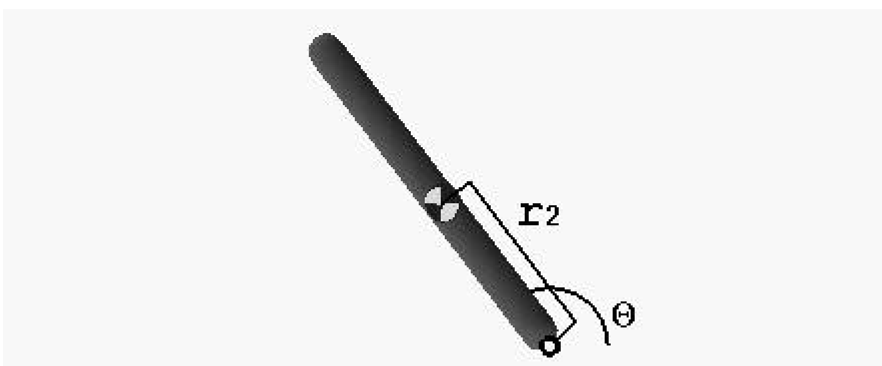
Cross Product $\mathbf{A} = \mathbf{B}\ \mathrm{x}\ \mathbf{C}$:

$\mathbf{A}_x = \mathbf{B}_y\,\mathbf{C}_z - \mathbf{B}_z\,\mathbf{C}_y \qquad \mathbf{A}_y = \mathbf{B}_z\,\mathbf{C}_x - \mathbf{B}_x\,\mathbf{C}_z \qquad \mathbf{A}_z = \mathbf{B}_x\,\mathbf{C}_y - \mathbf{B}_y\,\mathbf{C}_x$

Representing a set of equation in matrix form:

$$\begin{aligned} \mathbf{Y}_1 &= M_{11}\mathbf{X}_1 + M_{12}\mathbf{X}_2 \\ \mathbf{Y}_2 &= M_{21}\mathbf{X}_1 + M_{22}\mathbf{X}_2 \end{aligned} \quad \longrightarrow \quad \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$$

Relative Motion Equations: $\mathbf{a}_{cg} = \mathbf{a}_{end} + \mathbf{a}_{tangent} + \mathbf{a}_{normal}$



where: $\mathbf{a}_{cg}$ = translation acceleration at a segment's center of gravity
$\mathbf{a}_{end}$ = translation acceleration at a segment's end

$\mathbf{a}_{tangent} = \ddot{q}\ \mathrm{x}\ \mathbf{r}$ $\qquad$ $\mathbf{a}_{normal} = \dot{q}\ \mathrm{x}\ (\dot{q}\ \mathrm{x}\ \mathbf{r})$

and: $\dot{q}$ = angular velocity $\qquad$ $\ddot{q}$ = angular acceleration
$\mathbf{r}$ = vector from the segment end to the center of gravity

Trigonometric Identities: $\sin^2\theta + \cos^2\theta = 1$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\cos\phi$