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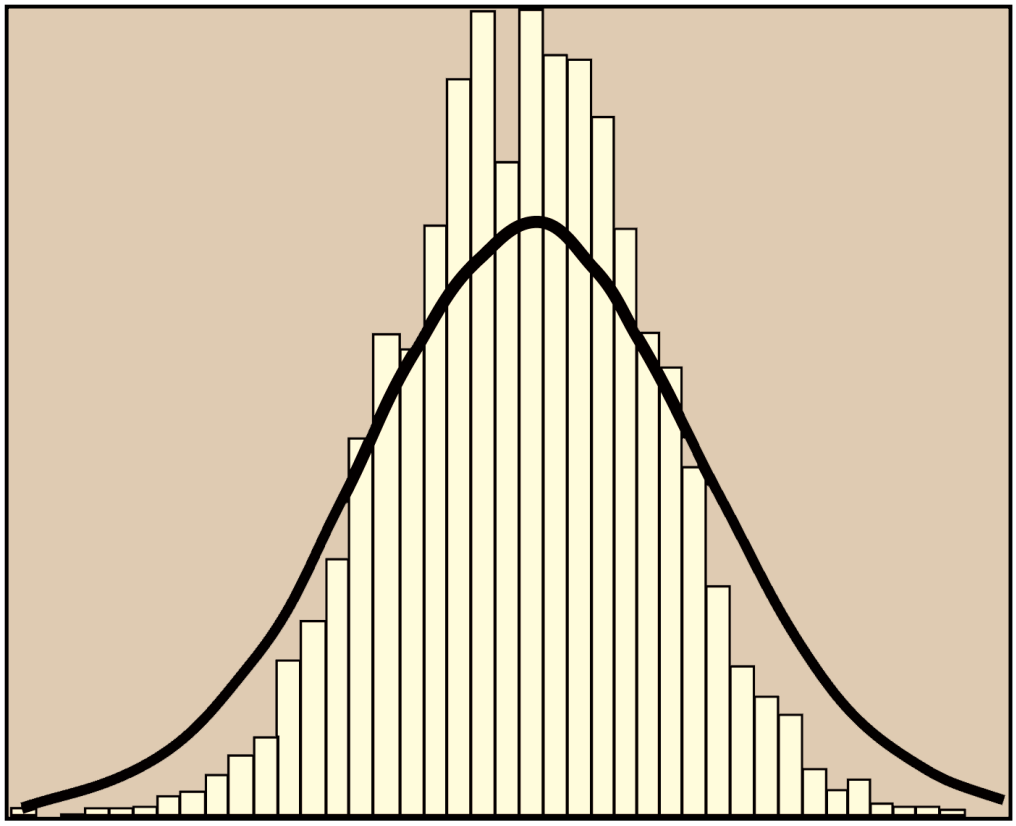
Steve P. Verrill
David W. Green
Victoria L. Herian



Abstract

In a series of papers Verrill, Green, and Herian have developed theory and a computer program to aid in the design and analysis of predictor sort experiments. In this paper we provide the mathematical justification for the power calculations that the predictor sort program implements.

Keywords: power calculations, predictor sort sampling, noncentral *F*

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Power Calculations in the Predictor Sort Computer Program

Steve P. Verrill, Mathematical Statistician
David W. Green, Supervisory Research General Engineer
Victoria L. Herian, Statistician
Forest Products Laboratory, Madison, Wisconsin

1 Introduction

In recent years wood strength researchers have replaced experimental unit allocation via random sampling with allocation via sorts based on non-destructive measurements of strength predictors such as modulus of elasticity and specific gravity. Although this procedure has the potential of greatly increasing experimental sensitivity, improperly implemented it can reduce sensitivity. Verrill (1993), Verrill and Green (1996), and Verrill (1999) address this problem. Verrill and others (1997) provide annotated input to and output from a computer program [1] — the "predictor sort" program — that implements some of the methods discussed in the 1996 paper. In this note we discuss the power calculations that are implemented in the predictor sort computer program.

2 Case 1 — Two Levels for One Factor

In the case of two levels for one factor, we first determine whether we can interpolate in the power tables obtained via simulation (tables 19–54, Verrill and Green, 1996). For interpolation in the tables to be possible, the sample size, the treatment difference divided by the coefficient of variation ratio, and the correlation between the predictor and the response must all lie within certain bounds. If they do not lie within the appropriate bounds, an approximate approach is taken. The reported power value is then

$$\text{Power} = \text{Prob}\left(F_{1,2k-2,\gamma^2} > F_{1,2k-2}(1-\alpha)\right)$$

(the probability that a noncentral F distribution with 1 numerator degree of freedom, $2k-2$ denominator degrees of freedom, and noncentrality parameter γ^2 lies above the $100(1-\alpha)$th percentile of a central F distribution with 1 numerator degree of freedom and $2k-2$ denominator degrees of freedom). Here k is the number of replicates for each of the two levels of the factor. The noncentrality parameter is

$$\gamma^2 = \left[(\Delta/\sigma_y) \Big/ \sqrt{2(1-\rho^2)/k}\right]^2$$

where Δ is the difference in the mean response for the two levels, σ_y is the variance of the response (for a fixed level), and ρ is the correlation between the predictor and the response. The CDFT and CDFFNC functions of DCDFLIB[2] were used to calculate the central F critical value and the noncentral F probability.

[1] This program can be run over the World Wide Web at `http://www1.fpl.fs.fed.us/ttweb.html`. This web page also provides links to LaTeX and postscript versions of this note's references.

[2] DCDFLIB is a public domain library of "routines for cumulative distribution functions, their inverses, and their parameters." It was produced by Barry Brown (`bwb@odin.mda.uth.tmc.edu`), James Lovato, and Kathy Russell of the Department of Biomathematics, M.D. Anderson Cancer Center, The University of Texas. The DCDFLIB source code can be obtained at `http://odin.mdacc.tmc.edu/anonftp/source.html`.

3 Case 2 — Multiple Factors/Multiple Levels

To explain our approach in the case of multiple factors and/or multiple levels, we use a combination of the notation in the appendix to Verrill (1993) and in section 4 of Verrill (1999).

Assume that the predictor variable and the variable of interest, Y, have a joint bivariate normal distribution with correlation ρ. Denote the variance of Y by σ_Y^2. Suppose that there are I blocks and F factors with $K_1, \ldots, K_F$ levels. Let the allocation of samples be as described in Section 1 of Verrill (1999). (For a multiple factor case, enough adjacent experimental units would be chosen at a time to provide one additional observation for each cell.) Let $\bar{Y}_{\cdot j_1 \cdot \ldots \cdot}$ be the standard estimator of mean response for the j_1th level of factor 1.

Approximate distribution of the test statistic numerator sum of squares when the null hypothesis does not hold

Suppose that we are interested in a test of the hypothesis that there is no difference in the effects of the K_1 levels of factor 1. (The argument is the same for the other factors.) The numerator sum of squares of the relevant F statistic is then

$$\text{NumSS} = \sum_{j_1=1}^{K_1} (I \times K_2 \times \ldots \times K_F)(\bar{Y}_{\cdot j_1 \cdot \ldots \cdot} - \bar{Y}_{\cdot \ldots \cdot})^2$$

where

$$Y_{i j_1 \ldots j_F} = E(Y_{i j_1 \ldots j_F}) + \sigma_Y \left(\rho X_{i j_1 \ldots j_F} + \sqrt{1-\rho^2} Z_{i j_1 \ldots j_F} \right),$$

the $X_{i j_1 \ldots j_F}$'s, $j_1 \in \{1, \ldots, K_1\}$, $\ldots$, $j_F \in \{1, \ldots, K_F\}$, are a randomization of the ith group of order statistics from $I \times K_1 \times \ldots \times K_F$ iid N(0,1)'s, the $Z_{i j_1 \ldots j_F}$'s are iid N(0,1), and the X's and Z's are independent. Thus

$$\begin{aligned}
\text{NumSS} &= \sum_{j_1=1}^{K_1} (I \times K_2 \times \ldots \times K_F) \\
&\quad \times \left(\mu_{j_1 \cdot \ldots \cdot} + \sigma_Y \left(\rho \bar{X}_{\cdot j_1 \cdot \ldots \cdot} + \sqrt{1-\rho^2} \bar{Z}_{\cdot j_1 \cdot \ldots \cdot} \right) - \mu_{\cdot \ldots \cdot} - \sigma_Y \left(\rho \bar{X}_{\cdot \ldots \cdot} + \sqrt{1-\rho^2} \bar{Z}_{\cdot \ldots \cdot} \right) \right)^2 \\
&= \sum_{j_1=1}^{K_1} (I \times K_2 \times \ldots \times K_F) \sigma_Y^2 \rho^2 (\bar{X}_{\cdot j_1 \cdot \ldots \cdot} - \bar{X}_{\cdot \ldots \cdot})^2 \qquad (1) \\
&\quad + 2 \sum_{j_1=1}^{K_1} (I \times K_2 \times \ldots \times K_F) \sigma_Y \rho (\bar{X}_{\cdot j_1 \cdot \ldots \cdot} - \bar{X}_{\cdot \ldots \cdot}) \\
&\qquad \times \left(\sigma_Y \sqrt{1-\rho^2} (\bar{Z}_{\cdot j_1 \cdot \ldots \cdot} - \bar{Z}_{\cdot \ldots \cdot}) + \mu_{j_1 \cdot \ldots \cdot} - \mu_{\cdot \ldots \cdot} \right) \\
&\quad + \sum_{j_1=1}^{K_1} (I \times K_2 \times \ldots \times K_F) \sigma_Y^2 (1-\rho^2) \\
&\qquad \times \left(\bar{Z}_{\cdot j_1 \cdot \ldots \cdot} - \bar{Z}_{\cdot \ldots \cdot} + (\mu_{j_1 \cdot \ldots \cdot} - \mu_{\cdot \ldots \cdot}) / \left(\sigma_Y \sqrt{1-\rho^2} \right) \right)^2
\end{aligned}$$

where $\mu_{j_1 \cdot \ldots \cdot} = E(\bar{Y}_{\cdot j_1 \cdot \ldots \cdot})$ and $\mu_{\cdot \ldots \cdot} = E(\bar{Y}_{\cdot \ldots \cdot})$. Now by the same reasoning as that used in section A.1.1 of Verrill (1993), the first term in (1) converges in probability to zero.

The last term in (1) is of the form

$$(I \times K_2 \times \ldots \times K_F)\sigma_Y^2(1-\rho^2) \sum_{j_1=1}^{K_1} (B_{j_1} - \bar{B})^2$$

where

$$B_{j_1} \sim \mathrm{N}\Big(\mu_{j_1 \cdot \ldots \cdot} \Big/ \Big(\sigma_Y \sqrt{1-\rho^2}\Big), 1/(I \times K_2 \times \ldots \times K_F)\Big).$$

Thus

$$\mathrm{NumSS} \approx \sigma_Y^2(1-\rho^2) \sum_{j_1=1}^{K_1} (C_{j_1} - \bar{C})^2$$

where

$$C_{j_1} \sim \mathrm{N}(\sqrt{I \times K_2 \times \ldots \times K_F}\mu_{j_1 \cdot \ldots \cdot} \Big/ \Big(\sigma_Y \sqrt{1-\rho^2}\Big), 1)$$

and the C_{j_1}'s are independent. It is well known that in this case $\sum_{j_1=1}^{K_1}(C_{j_1} - \bar{C})^2$ is distributed as a noncentral χ^2 with $K_1 - 1$ degrees of freedom and noncentrality parameter

$$\gamma^2 = \sum_{j_1=1}^{K_1} (\mu_{j_1 \cdot \ldots \cdot} - \mu_{\cdot \ldots \cdot})^2 \times (I \times K_2 \times \ldots \times K_F)/(\sigma_Y^2(1-\rho^2)) \tag{2}$$

Finally, by the Cauchy–Schwarz inequality and the work above, the second term in (1) is asymptotically negligible with respect to the third term in (1). Thus for larger I we have

$$\mathrm{NumSS} \approx \sigma_Y^2(1-\rho^2)\chi^2_{K_1-1,\gamma^2}$$

where γ^2 is given by (2).

Approximate distribution of the denominator sum of squares in the blocked case

The distributions of the blocked and unblocked F denominators discussed in the appendix to Verrill (1993) do not change under a non-null hypothesis. By following the argument made in appendix A.1.3 of Verrill (1993) one can show that in the blocked case, for larger sample sizes, the distribution of the denominator sum of squares is approximately $\sigma_Y^2(1-\rho^2)$ times a central χ^2 with $I \times K_1 \times \ldots \times K_F - (I + K_1 + \ldots + K_F - F)$ degrees of freedom, and it is approximately independent of the numerator sum of squares.

Approximate power [3]

We conjecture from the asymptotic results that small sample powers can be approximated in the blocked case by treating the blocked statistic as a noncentral F statistic with $K_1 - 1$ numerator degrees of freedom, $I \times K_1 \times \ldots \times K_F - (I + K_1 + \ldots + K_F - F)$ denominator degrees of freedom, and noncentrality parameter given by (2).

[3] The entry point to the power programs on the web is

`http://www1.fpl.fs.fed.us/ttweb.html`

Sample HTML, Perl, and FORTRAN code to perform these power calculations can be found at

`http://www1.fpl.fs.fed.us/power.code.html`

The power value reported by the predictor sort program is thus

$$\text{Power} = \text{Prob}\left(F_{\text{DOFN,DOFD},\gamma^2} > F_{\text{DOFN,DOFD}}(1-\alpha)\right)$$

(the probability that a noncentral F distribution with DOFN numerator degree of freedom, DOFD denominator degrees of freedom, and noncentrality parameter γ^2 lies above the $100(1-\alpha)$th percentile of a central F distribution with DOFN numerator degree of freedom and DOFD denominator degrees of freedom). Here

$$\begin{aligned} \text{DOFN} &= K_1 - 1 \\ \text{DOFD} &= I \times K_1 \times \ldots \times K_F - (I + K_1 + \ldots + K_F - F) \end{aligned}$$

and γ^2 is given by (2). The CDFF and CDFFNC functions of DCDFLIB were used to calculate the central F critical value and the noncentral F probability.

Power simulations

Because the "asymptotic" power results are only approximate, the predictor sort computer program permits a user to check the asymptotic results with small sample simulations. The program generates bivariate normal random variables (predictor/response pairs) with a user-specified correlation. It then allocates the responses to the treatments via a predictor sort. It produces design matrices that are appropriate for "pooled," "paired," and analysis of covariance analyses (see Verrill and Green (1996)). It makes use of both null hypothesis response vectors and non-null response vectors (to obtain simulation estimates of both size and power). It then performs the pooled, paired, and ANOCOV analyses on the generated data to detect statistically significant results. This process is repeated for the number of trials specified by the user.

The program performs one further theoretical check on the simulations by calculating an analytical estimate of the power in the ANOCOV case. This estimate is the average of the theoretical power values obtained for all the trials. (The power of an ANOCOV will depend on the column composed of predictor values, and these change from trial to trial.)

The numerical techniques used to perform the simulations will be described in a later publication[4].

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[4] "Rolling Your Own: Linear Model Hypothesis Testing and Power Calculations via the Singular Value Decomposition"