The Theory of Bose–Einstein Condensation of Dilute Gases

Bose–Einstein condensates are an ideal testing ground for quantum field theory in real time and at finite temperatures—basic topics of great importance for diverse physical systems.

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Bose–Einstein condensation (BEC) has long been known to be a key element of macroscopic quantum phenomena such as superconductivity and superfluidity. BEC per se, however, eluded direct and unquestioned observation until 1995, when experimental groups produced condensates in dilute atomic alkali gases.[1]

The story of these BEC experiments, as recounted in the accompanying article by Wolfgang Ketterle (page 30), has many of the elements of a heroic fable. Success was founded on the ingenuity, skill, and determination of the heroes, but it was hastened by their acquisition of "magic weapons," such as laser cooling, and by serendipity, such as having favorable values of fundamental atomic collision parameters. The only thing missing is the proverbial happy ending, for BEC itself has turned out to be a magic weapon that has launched other ambitious new quests during the subsequent four years.

Difficult though it was at first to attain, BEC has been found to provide a robust and versatile platform for experiments on mesoscopic many-body physics.[2] It has pushed the *ultima* Thule of low-temperature physics into territory some five orders of magnitude colder than the millikelvin regime of the helium superfluids; offered novel perspectives on phenomena previously encountered only in those superfluids; and provided precise tests of some of the keystone theories of many-particle quantum systems. Moreover, it has led to the production of entirely new physical systems, such as mixed degenerate Fermi gases (see Physics Today, October 1999, page 17), and has stimulated visions of extraordinary applications, such as coherent amplification of matter waves and table-top tests of finite-temperature quantum field theory.

This article presents a current perspective on advances in the theoretical understanding of gaseous BEC from the standpoint of atomic, molecular, and optical (AMO) physics. In AMO physics, BEC is now perceived both as an enabling technology, yielding the same exquisite control of matter waves that is possible for light waves, and as a vibrant point of contact with other branches of physics. Much of the essence of BEC in trapped-atom systems is captured in concepts that are familiar to AMO physicists yet have rich parallels in condensed matter, statistical, and elementary particle physics. For example, the order parameter, introduced by Lev Landau as a unifying concept for understanding phase transitions, is manifested in dilute gas BEC as the condensate wavefunction (see the cover of this issue and figure 1), and it can be measured, photographed, and manipulated in the laboratory.

What are the special features of Bose–Einstein condensed atomic gases, and why are they worthy of the intense current interest? First and foremost, they are assemblies of particles in a condensate with mesoscopic quantum features. Gaseous condensates exhibit very different properties from those in liquid helium. For example, more than 99% of the alkali atoms are in the condensate at $T = 0$, in contrast to liquid helium, in which the fraction is only on the order of 10%. It is possible to examine the behavior of gaseous condensates directly in space and time over a wide range of conditions. The relaxation of BEC systems far from equilibrium can also be studied, including direct observation of condensate formation. Moreover, quantitative theoretical predictions can be made for comparison with such experiments.

Do trapped gases actually exhibit BEC?

In 1925, Albert Einstein identified a phase transition in the theory of an ideal quantum gas of particles, obeying Bose–Einstein statistics, which occurs when the de Broglie wavelength of characteristic thermal motions, $\lambda_{\text{dB}} = (2\pi\hbar^2/mk_\text{B}T)^{1/2}$, becomes comparable to the mean interparticle separation, $r = \rho^{-1/3}$. (Here, m is the particle mass, k_B is Boltzmann's constant, T is the absolute temperature, and ρ is the atom number density.) The criterion for condensation of a uniform gas in three dimensions is

$$\rho\lambda_{\text{dB}}^3 > 2.612. \tag{1}$$

When this condition is attained, the lowest state of the system acquires a macroscopic population, even if the temperature is sufficiently high to populate many other states.

In what sense can it be claimed that the current set of experiments exhibit the condensation phenomenon predicted by Einstein? Current experiments deal with confined systems of a *finite* number of particles. Furthermore, interatomic interactions play a leading role in the energetics of these systems, so the ensembles of atoms cannot be treated as the ideal gas considered by Einstein. However, it turns out that the BEC transition in trapped gases is remarkably robust—it is not spoiled by the presence of interactions even in the vicinity of the phase transition.

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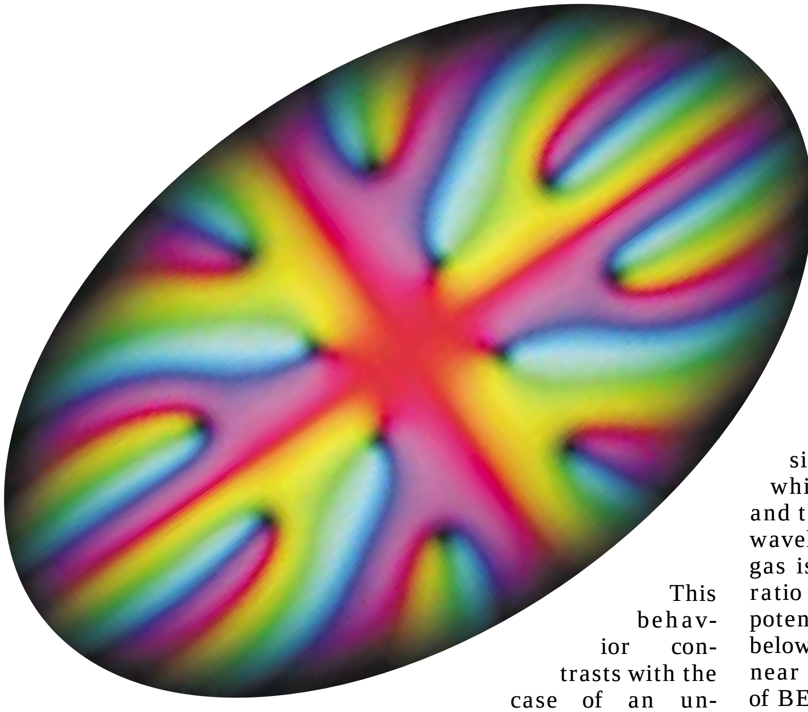

FIGURE 1. WAVEFUNCTION OF A BOSE–EINSTEIN CONDENSATE with 12 vortices present, calculated for a rotating condensate in an anisotropic trap. In this top view of the condensate midplane, the brightness is proportional to the amplitude of the wavefunction, and the color represents the wavefunction phase. The vortices pierce the figure through the black holes. (Courtesy of David L. Feder and Peter Ketcham, NIST.)

This behavior contrasts with the case of an untrapped, homogeneous gas, which enters a regime of critical fluctuations. We begin the discussion of this key question with a review of an ideal gas in a confining potential.

For a system of N noninteracting atoms in a spherically symmetric harmonic oscillator trapping potential with angular frequency ω, the critical temperature T_0 for the BEC transition is given[3]—in the thermodynamic or large-N limit—by $k_B T_0 = (N/\zeta(3))^{-1/3}\hbar\omega$, where $\zeta(3) \approx 1.202$ is the Riemann zeta function. Values of T_0 for the systems of current interest range from 10^{-7} to 10^{-4} K. For $T < T_0$, the number of condensate atoms, N_0, is given by

$$N_0 = N\left[1 - \left(\frac{T}{T_0}\right)^3\right]. \quad (2)$$

This expression for a trapped gas differs from the corresponding result for a homogeneous gas, which has an exponent of $^3/_2$. Figure 2 shows this thermodynamic result for large N, compared with the calculated condensate fractions that include only a finite number of atoms and both finite-size and interaction effects. The effect of finite size in trapped condensates is to yield transition temperatures that are not precisely defined, unlike the sharp transition in the thermodynamic limit. This issue has recently been addressed in considerable detail, and has provoked fresh debate over the nature of fluctuations in, and the applicability of standard ensembles to, the statistical mechanics of mesoscopic systems.[4]

The similarity of the curves in figure 2 emphasizes the underlying validity of the ideal-gas picture, which is due to the extremely low temperature of the gas and the consequent enormous size of the atoms' de Broglie wavelength. The de Broglie waves overlap even when the gas is still very dilute. The measure of diluteness is the ratio a/r of the characteristic range of the interaction potential, expressed by the scattering length a (discussed below), to the mean interparticle separation r. For ratios near 1, as is the case in liquid helium-4, the simple picture of BEC fails completely. For alkali gases of current interest, a/r is about 0.01.

Those familiar with the theory of critical phenomena might still be surprised that the transition region does not appear to be influenced by the presence of interactions. The presence of a trapping potential modifies the density of states at low energies from that of a homogeneous gas, and the ideal-gas theory works right through the transition. The occupation of low-energy states near the transition point spoils pure BEC in uniform gases. In the future, as larger and larger trapped condensates are made, the critical region will reemerge. This situation may lead to condensates with "phase domains"—regions in the condensate with differing wavefunction phases—which is an important issue for the phase coherence of matter-wave sources.

The interactions in a trapped gas shift in the transition temperature by a few percent for a typical experimental situation. Away from the transition region, the effects of interactions are extremely important in the dynamics of trapped condensates. Even for condensates as small as a few thousand atoms, the total energy of the condensate is made up of comparable contributions from the external trap potential and the atomic pair interactions, and the collective quasiparticle excitation spectrum of the condensate differs significantly from the single-particle excitation spectrum of the ideal gas.

Condensates at zero temperature

The current crop of gaseous Bose–Einstein condensates are confined, dilute, weakly interacting systems of cold bosonic (integer spin) atoms. For the dilute conditions of

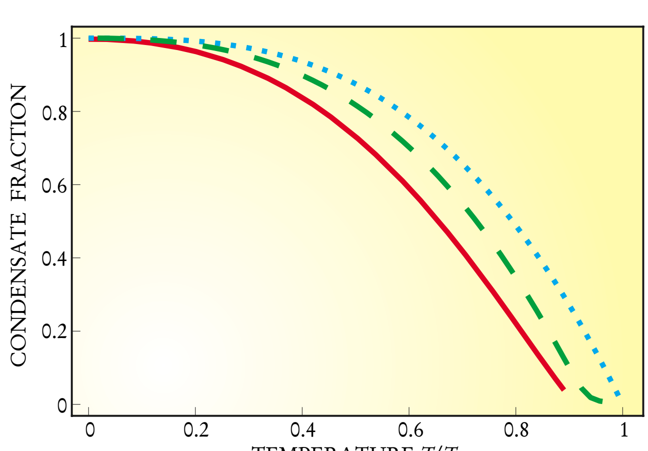

FIGURE 2. THE CONDENSATE FRACTION as a function of temperature is affected by both finite-size and interaction effects. Here are the predictions of three different models for the fraction of atoms that is in the condensate when the JILA trap is loaded with 2000 rubidium-87 atoms. The dotted line shows the thermodynamic result of equation 2. The dashed line is the result of incorporating finite-size effects, distributing 2000 atoms according to the Bose–Einstein distribution. The Popov theory (solid red line) includes both a finite number of atoms and atom–atom interactions, and gives very good agreement with observed condensate fractions.

these systems, the electronic degrees of freedom are frozen out and the atoms may be treated as discrete interacting particles. Whether Bose–Einstein or Fermi–Dirac statistics apply then depends upon the total number of spin-$^1/_2$ fermions (electrons plus nucleons) in the atom, so that the alkali isotopes with an odd number of nucleons are bosons.

At zero temperature, a condensate is a system of N_0 particles all occupying the same single-particle quantum state. In the simplest view of such a system, its many-body wavefunction can be written as the N_0-fold product of one single-particle, "condensate" wavefunction, $\psi(\mathbf{r}, t)$. This product is simply a Hartree many-body wavefunction (one can even term it a Hartree–Fock wavefunction since it is automatically symmetric under particle interchange). If the atoms did not interact, ψ would satisfy a single-atom, time-dependent Schrödinger equation. But how do we determine the condensate wavefunction when interactions are present?

Condensate atoms interact by means of binary collisions. Since the atoms are extremely cold, only head-on, or s-wave collisions are important; and because the gas is dilute, the interaction can be modeled by a zero-range potential whose strength is given by the s-wave scattering length a. (See the box on page 41 for more details.) In this case, each atom feels an additional potential due to the mean field of all the other atoms present, and this potential, proportional to the local atomic density, can be included in the Schrödinger equation to account for atom–atom interactions. The result is the nonlinear Schrödinger, or Gross–Pitaevskii (GP), equation

$$i\hbar \frac{\partial \psi}{\partial t} = H_0 \psi(\mathbf{r}, t) + N_0 U_0 |\psi(\mathbf{r}, t)|^2 \psi(\mathbf{r}, t), \qquad (3)$$

where the Hamiltonian H_0 includes the kinetic energy and $V_{\rm trap}$, the confining potential of the trap—typically a harmonic oscillator potential. The coefficient of the nonlinear term is given by $U_0 = 4\pi\hbar^2 a/m$, where m is the mass of a condensate atom. Note that when a is positive, the condensate atoms repel each other, and when a is negative, they attract.

When the condensate has a large number of atoms, the time-independent GP equation admits a simple solution. With the ansatz $\psi(\mathbf{r}, t) = e^{-i\mu t/\hbar}\phi(\mathbf{r})$, where μ is the chemical potential (the energy required to add one more atom to the condensate), the left side of equation 3 becomes $\mu\psi(\mathbf{r}, t)$. When the nonlinear interaction energy term is much greater than the kinetic energy term—the so-called Thomas–Fermi limit—one can neglect the kinetic energy and obtain an algebraic solution for the density profile of the condensate:

$$|\phi(\mathbf{r})|^2 \approx \frac{\mu - V_{\rm trap}(\mathbf{r})}{N_0 U_0}, \qquad (4)$$

wherever the right-hand side is positive, and zero otherwise. The value of μ is determined by normalizing ϕ. The density profile for a typical experimental condensate is thus an inverted paraboloid when the confining potential is harmonic. For the condensates currently being produced, typically with 10^6 or more atoms, the Thomas–Fermi approximation works very well.

Validity of the Gross–Pitaevskii equation

Two diverse examples demonstrate the usefulness of the GP equation. The first concerns the dynamics of a condensate when the trapping potential is turned off. In most of the experimentally realized condensates, the scattering length is positive and the atoms repel each other. Hence, when the trap is removed, the condensate expands ballistically. Experimentalists Eric Cornell, Carl Wieman, and their coworkers at JILA in Boulder, Colorado, used this expansion to obtain a condensate large enough to be easily imaged.[1] JILA theorists Murray Holland and John Cooper modeled this experiment by using the GP equation to calculate the evolution of a condensate after the trap had been turned off.[5] Figure 3 compares their calculated image of the density profile of the expanded condensate with the experimental image obtained in the original JILA experiment. The agreement was better than 5% and there were no adjustable parameters in the calculation.

Another important early application of the time-dependent GP equation was the prediction of condensate excitation spectra, for it showed how the energies and wavefunctions of quasiparticle excitations could be determined by experiment in a straightforward and readily visualized fashion. When a condensate is weakly disturbed by a sinusoidal perturbation of the correct symmetry, it will oscillate strongly if the frequency of the disturbance matches one of the condensate's characteristic frequencies. These excitation frequencies can be theoretically determined by examining the frequencies of small oscillations around a stationary solution of the GP equation. In other words, the equations that predict these frequencies

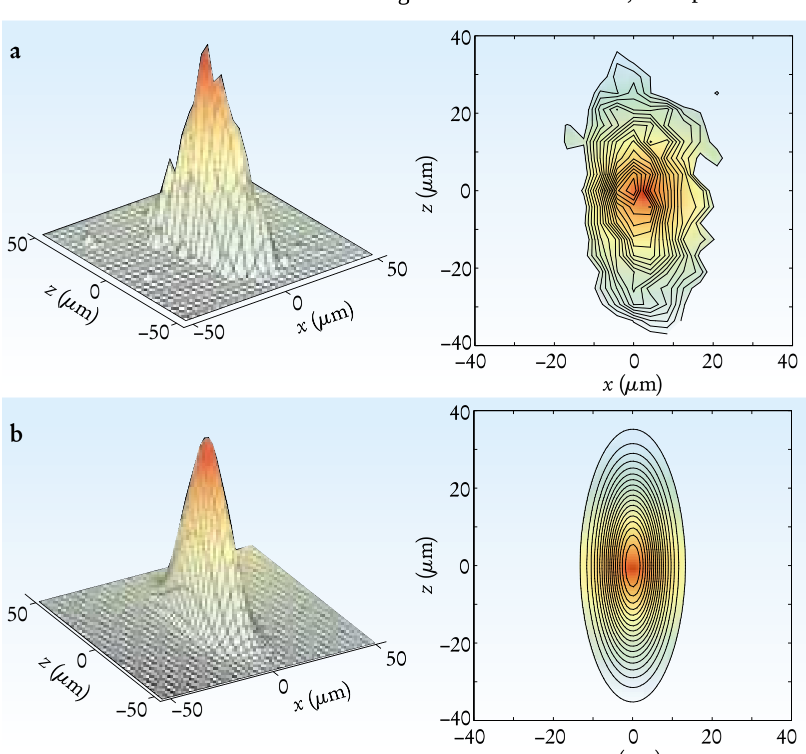

FIGURE 3. DENSITY PROFILE of a Bose–Einstein condensate. **(a)** In JILA's original observation of a Bose–Einstein condensate, the magnetic trap was turned off, the condensate allowed to expand, and the density profile of the condensate measured after 60 ms. **(b)** The predictions of the Gross–Pitaevskii equation for the same conditions, as calculated by Murray Holland and John Cooper (JILA). (Courtesy of Murray Holland.)

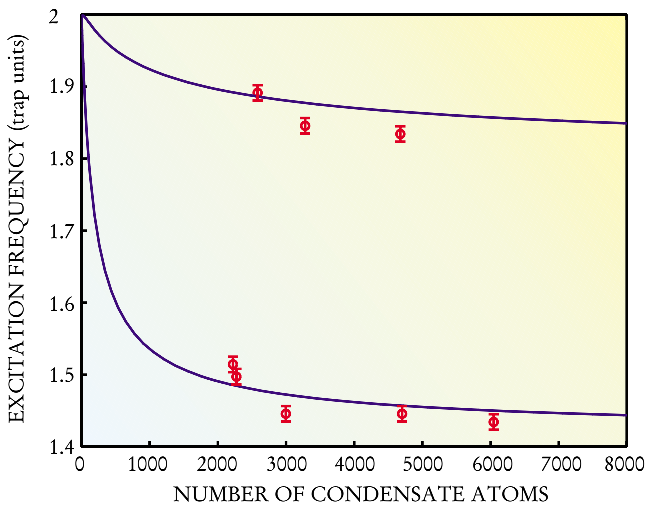

FIGURE 4. COLLECTIVE EXCITATION FREQUENCIES of a rubidium-87 condensate (red points), as measured in the JILA trap, compared with the predictions of zero-temperature linear-response theory (blue lines) for two excitations of different symmetry. The top curve is a radial "breathing mode" with zero units of angular momentum around the trap symmetry axis. The bottom curve is a quadrupole mode having two units of angular momentum around the trap axis.

can be found by performing a linear-response analysis on the GP equation. In this procedure, one starts with a static solution to the time-independent GP equation

$$H_0\psi_0(\mathbf{r}) + N_0U_0|\psi_0(\mathbf{r})|^2\,\psi_0(\mathbf{r}) = \mu\psi_0(\mathbf{r}), \tag{5}$$

where μ is again the chemical potential. Adding a small sinusoidal perturbation to the potential yields two coupled equations that can be solved for the quasiparticle modes and frequencies:

$$\begin{aligned} \mathcal{L}u_\lambda(\mathbf{r}) + N_0U_0(\psi_0(\mathbf{r}))^2\,v_\lambda(\mathbf{r}) &= \hbar\omega_\lambda\,u_\lambda(\mathbf{r}) \\ \mathcal{L}v_\lambda(\mathbf{r}) + N_0U_0(\psi_0^*(\mathbf{r}))^2\,u_\lambda(\mathbf{r}) &= -\hbar\omega_\lambda\,v_\lambda(\mathbf{r}), \end{aligned} \tag{6}$$

where $\mathcal{L} \equiv H_0 + 2N_0U_0|\psi_0(\mathbf{r})|^2 - \mu$. Here, $u_\lambda(\mathbf{r})$ and $v_\lambda(\mathbf{r})$ describe the normal modes of the quasiparticle excitations, and ω_λ are the frequencies of collective excitations of the condensate. Equations 5 and 6 are the same as what Nikolai Bogoliubov derived by a different means 50 years ago for a weakly interacting, dilute Bose gas—a system that didn't even exist then.

Figure 4 shows the predictions from the Bogoliubov equations for the excitation frequencies observed in the JILA rubidium-87 condensate.[6] In spite of no adjustable parameters, the calculations agreed with the measured frequencies at about the 2% level. These comparisons were performed for relatively small condensates ($N_0 \sim 10^4$). Wolfgang Ketterle and his coworkers at MIT have measured[7] the excitation frequencies of condensates with $N_0 \sim 10^6$. Sandro Stringari at the University of Trento (Italy) obtained an analytic expression for the excitation frequencies in this large-N_0 regime, in which the frequencies are independent of the scattering length and of the number of atoms in the condensate.[8] The agreement between theory and experiment for these large condensates was again at a few percent.

The above treatment applies only to condensates with positive scattering lengths. A negative scattering length in a homogeneous gas implies complex excitation frequencies and instability to collapse. Physically, one can lower the energy of a gas having effectively attractive interactions by creating a more dense region within it. It was therefore generally assumed that BEC could be seen only in condensates with positive scattering lengths. However, we found that, for small numbers of atoms, there are stable solutions of the GP equation for trapped condensates with negative scattering lengths.[9] The zero-point energy provided by the trap balances the attraction of the atoms for each other and prevents the collapse. This predicted stability has been confirmed by Randy Hulet's group at Rice University for condensates of lithium-7 atoms, which have a negative scattering length. Stable condensates with $a < 0$ thus exist only for trapped gases, and there is no counterpart for this type of BEC in a homogeneous gas.

The existence of condensates with negative scattering lengths has led to a detailed examination of the way in which the attractive interactions affect the condensates. The effects include possible dynamical processes, such as the collapse of a condensate by macroscopic tunneling to a more compressed state. A related possibility in an ultracold Fermi gas—forming the analog of a superconducting state through a Bardeen-Cooper-Schrieffer (BCS) transition—is currently a matter of active investigation.

Condensates at finite temperature

The standard mean-field theory that describes a finite-temperature trapped gas in thermal equilibrium is the Hartree-Fock-Bogoliubov (HFB) theory.[10] The structure of the HFB equations is similar to that of the Bogoliubov equations (equations 5 and 6) for zero temperature. The main differences are the inclusion of a temperature-dependent density of noncondensate atoms, and the presence of the so-called anomalous density, which accounts for the correlations between the atoms and is equivalent to the pairing field of BCS theory that produces the gap in

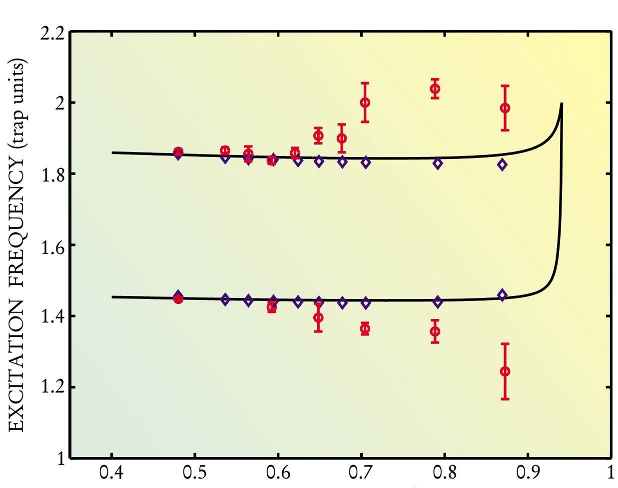

FIGURE 5. COLLECTIVE EXCITATION FREQUENCIES of condensates provide a sensitive test of finite-temperature, many-body theory. The red circles are the collective excitation frequencies measured[11] at finite temperatures in the same trap as in figure 3. The blue diamonds are the predictions of the Popov theory. The solid black curves, which match well with the Popov theory, are the excitation frequencies for zero-temperature condensates with the same number of atoms as in the finite-temperature system. Although the Popov theory makes accurate predictions for condensate fractions and specific heats, it fails to reproduce experimental collective excitation frequencies at temperatures near the Bose–Einstein transition temperature.

superconductivity. If the anomalous density is neglected, then one has the so-called Popov approximation, which has been extensively used in the study of BEC at finite temperatures. In the Popov approximation, the trapped gas can be thought of as a condensate plus a thermal ideal gas. Condensate fractions and specific heats calculated using the Popov equations have matched well with experiment.[2]

The success of the Popov theory for condensate fractions and specific heats lies in the fact that these quantities depend on the entire quasiparticle excitation spectrum. Individual quasiparticle frequencies provide a more sensitive test of finite-temperature theory. Such excitation frequencies were measured at JILA as a function of temperature and are shown in figure 5, along with our own calculations using Popov theory.[11] The agreement between theory and experiment is again at about 5%, for $T/T_0 \leq 0.65$ or a measured condensate fraction of at least 50%. For higher temperatures, however, the Popov theory clearly does not match the data. This discrepancy remains an open question, and its resolution has been an active area of recent study. One clear difficulty with the Popov theory is the assumption that the noncondensate part of the trapped gas is static, which has not been the case in excitation measurements performed thus far.

The use of Popov theory was initially motivated by the concern that the condensate excitation spectrum be "gapless"—that is, that there be a zero-frequency excitation. A gapless theory is expected to give a better account of low-energy elementary excitations than a theory having a gap. In a uniform gas, long-wavelength excitations having an energy that vanishes with vanishing wavenumber are called Goldstone bosons and always arise in field theory when a continuous symmetry of the field—here the phase of the condensate wavefunction—is spontaneously broken. In a trapped gas, which has discrete excitations, the Goldstone mode is a zero-frequency solution of the Popov (or Bogoliubov at $T = 0$) equations. Such a mode is always a solution of the Popov equations, but no such mode exists for the full HFB equations due to the anomalous density. Attempts to include the anomalous density perturbatively in the Popov equations have only partly succeeded:[12] Although calculations for the quadrupole mode (the bottom curve in figure 5) agreed with experimental results, the observed behavior of the mode with zero angular momentum could not be explained.

The phase of the condensate

There has been a great deal of discussion about the nature of phase and its relation to spontaneous symmetry breaking in mesoscopic systems. In the case of infinite systems, there is a rigorous basis for the concept of spontaneous symmetry breaking, in which a continuous aspect of a system, such as its phase, adopts one value although all values are equally acceptable and likely. For a trapped gas, one must consider what phase means for a finite number of particles.

One can measure the relative phase of two condensates by observing their interference where they overlap.[13] As is known from the quantum theory of phase and its measurement, the relative phase between two condensates, each with a definite number of particles in it, is not defined—number and phase obey the uncertainty relationship $\Delta N \Delta \phi \geq 1$. An interference pattern is nevertheless obtained experimentally. As long as which of the two condensates contributed each atom in the interference region is not determined, then after the arrival of a few atoms, one cannot know the number of atoms in either of the source condensates. This uncertainty is, after all, the condition for interference and is precisely what is needed

Ultracold Interactions

Bose–Einstein condensation is reached when the interparticle separation is comparable to the de Broglie wavelength of the atoms. For evaporatively cooled gases, the de Broglie wavelength of the atoms is enormous, compared to the range of the interatomic forces. We can therefore model binary scattering using an effective contact interaction: $V(\mathbf{r} - \mathbf{r}') = U_0\delta(\mathbf{r} - \mathbf{r}')$. Here, U_0 is given in terms of the binary s-wave scattering length a by $U_0 = 4\pi\hbar^2 a/m$, which appears in equation 3, the Gross–Pitaevskii equation. This interaction gives the exact low-energy scattering amplitude ($-a$) when used in the simplest, first-order perturbation theory approximation (the Born approximation).

To see how the contact interaction changes the energy of the gas, one can consider the relative wavefunction of a pair of alkali atoms scattering off one another. For ultralow scattering energies, the effect of the interatomic potential is equivalent to that of a hard sphere of radius a. When the scattering energy is zero, the relative wavefunction has the form $\phi(r) = \chi(1 - a/r)$. Here, a is the scattering length and χ is the asymptotic value of the wavefunction. Written in this way, the zero-energy wavefunction clearly has a node at a. (The above wavefunction is valid only outside the range of the atomic potential; for smaller distances, the wavefunction depends on the details of the interatomic potential.)

In the dilute gas, the scattering length provides all of the information needed to calculate the change in the energy of the gas due to the interactions between the particles. In the limit of low scattering energies, this additional energy is stored in the increased kinetic energy of the particles produced by the boundary condition of a node at $r = a$. This extra kinetic energy in the wavefunction is given by

$$\int_a^\infty dr\,(4\pi r^2)\,\frac{\hbar^2}{m}\left\{\chi\nabla\left[1-\frac{a}{r}\right]\right\}^2 = U_0\chi^2.$$

If one takes χ^2 as the density of the other particles, one obtains the needed expression for the energy of one particle in the presence of others.

At ultralow temperatures, the scattering length can be much larger (typically ten to a hundred times bigger) than the hard-core size of the atoms assumed in kinetic theory for room-temperature atoms. Because of this large scattering length, collisional relaxation to thermal equilibrium is relatively quick compared to the rate at which atoms are lost from the trap. For the condensates made thus far, the scattering lengths are still very small compared to the distance between atoms—a required condition for the gas to be weakly interacting or, equivalently, for the condensate fraction to be large. Thus, for the alkali condensates,

$$\lambda_{\mathrm{dB}} \gg \rho^{-1/3} \gg a,$$

where ρ is the peak density of the trapped gas and λ_{dB} is the de Broglie wavelength of the atoms. The scattering length can also be negative when there is an effectively attractive interaction between the atoms.

To estimate the scattering length, one needs very precise knowledge of the interatomic potential. For hydrogen, a can be calculated directly from molecular quantum mechanics. For alkali atoms, the estimation of a has relied on the development of new spectroscopic methods, particularly photoassociation spectroscopy and trap-loss spectroscopy. The theoretical and experimental technologies that now exist have yielded a very precise understanding of the interactions between ultracold atoms,[16] which provides a crucial advantage in analyzing assemblies of Bose–Einstein condensed atoms. The scattering length can be accurately determined and not treated as an adjustable parameter.

to produce a state with a well-defined relative phase. The same interference pattern is obtained if one assumes that each condensate has a well-defined phase before the measurement. The relative phase becomes more and more precisely defined just as the relative number distribution between the two condensates becomes more and more uncertain. Each condensate evolves into a superposition of number states with a Poissonian probability distribution. The condensate is thus in a coherent state, analogous to the state of the field inside a laser cavity.

For the order parameter to have the simple $e^{-i\mu t/\hbar}$ time dependence, the condensate must be truly macroscopic. If the system is finite, then there will be a timescale over which the components of the coherent state get out of step. Whether this decoherence can be observed in an experiment is currently being studied.

BEC formation and laser action

There is an important link between laser action and the production of a condensate. The Bose–Einstein distribution arises from the detailed balance of collisions between pairs of particles—be they photons or bosonic atoms in a dilute gas. When degeneracy becomes important, the dependence of the collision rate on the occupation of final states must be retained—that is, there are stimulated collision processes. For lasers, it is the stimulated emission of a photon into a particular mode of the radiation field that causes laser action. For atoms in a trap, the stimulated scattering of atoms produces a buildup of population in the ground state. A quantitative theory of this behavior, quantum kinetic theory,[14] has been developed and used to predict the time evolution of a condensate formed after a rapid quench of the energy in a gas very close to condensation. Such studies have confirmed the critical role of Bose stimulation in the formation of a condensate.

What are some of the features of a laser mode that should be looked for in the behavior of a condensate? One is the reduction of fluctuations, compared to a thermal source. This reduction is manifest in two- and three-photon absorption by elements placed inside the laser cavity: Nonlinear absorption is very sensitive to fluctuations in the intensity of the laser mode. One can look for analogous behavior in a condensate in the rate of decay of the condensate due to two- or three-body collisions—processes equivalent to nonlinear absorption of the laser mode. For example, consider the case of three-body collisions in a little more detail. The local rate of decay depends on the average value of the cubed density, $\langle\rho^3\rangle$. For a thermal Bose gas, one expects fluctuations such that

$$\langle\rho^3(\mathbf{r})\rangle \sim 6\langle\rho(\mathbf{r})\rangle^3. \tag{7}$$

For a laser mode or a condensate, the corresponding result is

$$\langle\rho^3(\mathbf{r})\rangle \sim \langle\rho(\mathbf{r})\rangle^3. \tag{8}$$

Formally, this difference arises because the laser mode and the condensate are well represented by coherent states. One could thus attempt to verify that a condensate is like a laser—that is, in a coherent state—by measuring the three-body decay rate above and below the BEC transition. When Wieman, Cornell, and their coworkers at JILA performed this experiment,[15] they found precisely the difference one would expect if the condensate were in a coherent state.

BEC: Present and future

The preceding account describes developments that have occurred in the last few years since the pioneering experiments. At the present time, the subject is moving into a broad range of new directions. The study of multiple component condensates enables us to see the evolution (such as folding and unwinding) of more complex order parameters in space and time. Phase transitions in lower dimensional systems are now accessible: The recent triumph in the study of the superfluid transition in films of spin-polarized hydrogen on liquid helium will soon be complemented by experiments on thin sheets of condensed alkali atoms confined by laser fields. The theory of these confined lower-dimensional systems looks very rich indeed. Direct formation of vortices and solitons in trapped gases has now been achieved (see PHYSICS TODAY, November 1999, page 17), and the study of their nucleation and stability is under way. The dilute gases enable researchers to magnify the structure of these systems and subject their dynamics to direct scrutiny. The formation and evolution of structure (domains, vortices, and so forth) in quenched conditions is under way based on large-scale simulation. These studies of topological excitations are driving the development of new computational and visualization techniques that can meet the considerable challenge that the theory presents.

The use of condensates as a source of coherent matter waves has only just begun, and it heralds a new age for precision measurement based on coherent atomic optics. The theory of matter-wave coherence and entanglement for atom laser sources is yielding genuinely new insights into the structure of many-body systems. For those working in the field, the connections that have been established between previously disparate communities has been a cause for great excitement, and the opportunities for quantitative theory abound in this new and growing playground for condensed matter and AMO theorists.

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